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Micro-local, Non-linear, Resolution Analysis of Generalised Radon Transform Inversions in Anisotropic Media

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Sincerely,

Norman Bleistein

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ROBERT J. SILVERMAN

Micro-local, non-linear, resolution analysis of generalised Radon transform inversions in anisotropic media

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Abstract. The resolution analysis of generalized Radon Transform (GRT) inversion of seismic data is carried out in general anisotropic media. The GRT inversion formula is derived form the ray-Born approximation of the wave field for volume scatterers. However, by considering scattering surfaces in the resolution analysis, rather than parameter perturbations, we show that the inversion provides a reflectivity map and reflection/transmission coefficients as functions of scattering angles and azimuths. Those coefficients can be subjected to any type of Amplitude Versus Angle (AVA) or Amplitude Versus Offset (AVO) analysis. By applying the inversion to Kirchhoff approximate data rather than Born approximate data, we show that the output is actually linear in the reflection coefficients and, hence, a nonlinear function of the change in medium parameters across discontinuity surfaces—the reflectors of the medium.

1. Introduction

Current acquisition techniques take reflection-seismic measurements at increasingly larger scattering angles both for image enhancement and improved estimation of elastic parameters. To achieve a better resolution, however, one must account for the interplay between heterogeneity and anisotropy at different length scales. Also, to accommodate inversion of wide angle data, one has to take account of the nonlinear dependence of the reflected amplitude on changes in medium parameters, at least near the relevant reflectors in the configuration. We use the *generalised Radon transform* or GRT formulation to achieve these goals.

Our analysis begins with the ray-Born approximation of the direct scattering problem, which represents the scattered field as a volume integral. This representation is recast as a sum of surface integrals over the discontinuity surfaces of the medium parameters—the reflectors in the subsurface. Then the inversion of the linearised scattering problem is carried out. The resolution analysis of the linearised GRT inversion serves as the basis to arrive at a micro-local nonlinear formulation: its range of validity is extended by applying a stationary phase analysis with respect to migration dip, defined as the normalised gradient of total travel time along the

scattering characteristic. In this process, carrying out the GRT inversion on Kirchhoff-type data, an explicit adjustment of the inversion formula is found.

By identifying the data, for each image point, in common scattering-angle/azimuth gathers, the GRT inversion formula can be modified to obtain reflection/transmission coefficients at specular ray geometries. Any amplitude versus scattering angle—AVA—analysis can then be applied to those coefficients to derive information about the medium perturbation. Several parametrizations of this perturbation have been developed to reveal to which quantities the coefficients are sensitive.

The idea of estimating angle-dependent reflectivity has been developed by various authors; see for example Geoltrain and Chovet [1] and Lumley [2]. A more rigorous discussion of such an approach can be found in Bleistein *et al.* [3]. In our paper, we derive a GRT-based inversion procedure that accomplishes the goal of constructing full reflection—and possibly transmission—coefficients as functions of scattering angle and azimuth, in closed form.

GRT-type inversion formulas for the *linearised* inverse problem have been developed by Norton and Linzer [4], by Beylkin [5, 6, 7, 8], by Miller *et al.* [9, 10], by Beylkin *et al.* [11], and by Rakesh [12] for the acoustic case. The extension to the elastic case was discussed by Beylkin and Burridge [13], and anisotropy was considered by De Hoop *et al.* [14] and Spencer and De Hoop [15]. Inversion formulas aiming to estimate reflectivity rather than the medium perturbation were developed by Cohen and Bleistein [16], by Bleistein and Cohen [17], and by Bleistein [18, 19]. This paper brings the two approaches together. Discussion of the numerical implementation of GRT inversion procedures can be found in De Hoop and Spencer [20].

The GRT approach is a *high frequency* approach to inversion. There are two equally valid points of view about the utility of this method. First, for full bandwidth data, we obtain by this method only the most *singular part* of the solution to the inverse problem, since it is this part of the solution that is tied to the limit of frequency approaching infinity. On the other hand, as a practical matter, the typical bandwidth of the seismic inverse problem is such that the data can be viewed as *high frequency* data for most of the length and time scales of the geophysical model. Thus, numerical implementation of the derived inversion formulas produce useful results for seismic exploration.

Throughout the paper we have excluded the possibility of multipathing of the rays connecting the image point to a source and a receiver in the predefined background medium. Certain changes in the formula's are necessary in that case, which we will investigate in a separate paper.

2. The basic equations

2.1. Notation

First, we introduce some basic notation. Choose coordinates in the configuration according to

$$\begin{aligned}
\boldsymbol{x} = (x_1, x_2, x_3) &= \text{Cartesian position vector,} \\
\boldsymbol{s} = (s_1, s_2, s_3) &= \text{source point,} \\
\boldsymbol{r} = (r_1, r_2, r_3) &= \text{receiver point,} \\
t &= \text{time.}
\end{aligned}$$

The medium is described by

$$\begin{aligned}
\rho(\boldsymbol{x}) &= \text{density,} \\
c_{ijk\ell}(\boldsymbol{x}) &= \text{elastic stiffness tensor,}
\end{aligned}$$

while the wavefield is described by

$$\boldsymbol{u}(\boldsymbol{x},t) = (u_1(\boldsymbol{x},t), u_2(\boldsymbol{x},t), u_3(\boldsymbol{x},t)) = \text{displacement vector,}$$

and generated by a source distribution given by

$$\boldsymbol{f}(\boldsymbol{x},t) = (f_1(\boldsymbol{x},t), f_2(\boldsymbol{x},t), f_3(\boldsymbol{x},t)) = \text{body-force source density.}$$

In the remainder of the paper, we will employ the summation convention.

2.2. Governing wave equation

The displacement in a heterogeneous anisotropic medium satisfies the elastodynamic wave equation

$$\rho\, \partial_t^2 u_i - \partial_j (c_{ijk\ell} \partial_\ell u_k) = f_i \ , \tag{1}$$

with summation over repeated lower case indices, here and below. Let

$$\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{x}', t) = (G_{ip}(\boldsymbol{x}, \boldsymbol{x}', t)) \tag{2}$$

be the causal Green's tensor, which satisfies (cf. Eq.(1))

$$\rho\, \partial_t^2 G_{ip} - \partial_j (c_{ijk\ell} \partial_\ell G_{kp}) = \delta_{ip} \delta(\boldsymbol{x} - \boldsymbol{x}') \delta(t) \ , \quad G_{ip} = 0 \ \text{ for } \ t < 0 \ . \tag{3}$$

2.3. Asymptotic ray theory

Here, we summarize the formulation of anisotropic ray theory for the evaluation of the Green's tensor (see, e.g., Kendall *et al.* [21]). Let

$$\begin{aligned}
G_{ip}(\boldsymbol{x}, \boldsymbol{x}', t) \;=\; & \sum_N A^{(N)}(\boldsymbol{x}, \boldsymbol{x}')\, \xi_i^{(N)}(\boldsymbol{x}) \xi_p^{(N)}(\boldsymbol{x}') \delta(t - \tau^{(N)}(\boldsymbol{x}, \boldsymbol{x}')) \\
& + \ \text{terms smoother in } t \ .
\end{aligned} \tag{4}$$

In this equation, the arrival time $\tau^{(N)}$ and the associated polarization vector $\boldsymbol{\xi}^{(N)}$ satisfy

$$(\rho\,\delta_{ik} - c_{ijk\ell}(\partial_\ell \tau^{(N)})(\partial_j \tau^{(N)}))\,\xi_k^{(N)} = 0 \quad \text{(at all } \boldsymbol{x}) \,, \tag{5}$$

which implies the eikonal equation

$$\det(\rho\,\delta_{ik} - c_{ijk\ell}(\partial_\ell \tau)\,(\partial_j \tau)) = 0 \quad \text{(at all } \boldsymbol{x}) \,. \tag{6}$$

The polarization vectors are assumed to be normalised so that $\xi_i^{(N)}\xi_i^{(N)} = 1$. Define the slowness vector $\boldsymbol{\gamma}^{(N)}$ by

$$\boldsymbol{\gamma}^{(N)}(\boldsymbol{x}) = \nabla_{\boldsymbol{x}} \tau^{(N)}(\boldsymbol{x}, \boldsymbol{x}') \,. \tag{7}$$

Then, Eq.(6) constrains $\boldsymbol{\gamma}$ to lie on the sextic surface $\mathcal{A}(\boldsymbol{x})$ given by

$$\det(\rho\,\delta_{ik} - c_{ijk\ell}\gamma_\ell\gamma_j) = 0 \,. \tag{8}$$

$\mathcal{A}(\boldsymbol{x})$ consists of three sheets $\mathcal{A}^{(N)}(\boldsymbol{x})$, $N = 1, 2, 3$, each of which is a closed surface surrounding the origin. An individual sheet is also described by (cf. Eq.(5))

$$2\,\mathcal{H} = \rho - \xi_i c_{ijk\ell}\gamma_\ell\gamma_j\xi_k = 0 \,. \tag{9}$$

The scalar amplitudes $A^{(N)}$ must satisfy the transport equation

$$\partial_j(c_{ijk\ell}\xi_i^{(N)}\xi_k^{(N)}\,(A^{(N)})^2\,\partial_\ell\tau^{(N)}) = 0 \,, \tag{10}$$

where N, again, indicates the mode of propagation, that is, the sheet of the slowness surface on which the corresponding slowness vector lies.

The characteristic or group velocities $\boldsymbol{v}^{(N)}$ are normal to $\mathcal{A}^{(N)}(\boldsymbol{x})$ at $\boldsymbol{\gamma}^{(N)}$ and satisfy

$$\boldsymbol{v}^{(N)} \cdot \boldsymbol{\gamma}^{(N)} = 1 \,; \quad \boldsymbol{v} = \left.\frac{\nabla_{\boldsymbol{\gamma}}\mathcal{H}}{\boldsymbol{\gamma}\cdot\nabla_{\boldsymbol{\gamma}}\mathcal{H}}\right|_{\mathcal{H}=0} \,, \tag{11}$$

see Eq.(9). The normal or phase speeds are given by

$$V^{(N)} = \frac{1}{|\boldsymbol{\gamma}^{(N)}|} \,. \tag{12}$$

The unit phase direction follows as

$$\boldsymbol{\alpha}^{(N)} = V^{(N)}\,\boldsymbol{\gamma}^{(N)} \,.$$

From Eq.(11) it follows that

$$V^{(N)} = |\boldsymbol{v}^{(N)}|\,\cos\chi^{(N)} \,, \tag{13}$$

where $\chi^{(N)}$ is the angle between $\boldsymbol{v}^{(N)}$ and $\boldsymbol{\gamma}^{(N)}$.

The amplitudes can be expressed in terms of certain Jacobians,

$$A = \frac{1}{4\pi[\rho(\boldsymbol{x})\rho(\boldsymbol{x}')\mathcal{M}]^{1/2}} \quad \text{with} \quad \mathcal{M} = \frac{|\boldsymbol{v}(\boldsymbol{x}')|V(\boldsymbol{x}) \left|\dfrac{\partial \boldsymbol{x}}{\partial q_1} \wedge \dfrac{\partial \boldsymbol{x}}{\partial q_2}\right|_{\boldsymbol{x}}}{\left|\dfrac{\partial \boldsymbol{\gamma}}{\partial q_1} \wedge \dfrac{\partial \boldsymbol{\gamma}}{\partial q_2}\right|_{\boldsymbol{x}'}} \,, \tag{14}$$

in which A, $\mathcal{M}$, $\boldsymbol{v}$ and V carry the superscript (N). Here, (q_1, q_2) parameterize the rays originating from the source. One can verify that the dimension of A is [time]^2 $\times$ [mass]$^{-1}$, which upon multiplication by force, with dimensions [mass] $\times$ [length] $\times$ [time]$^{-2}$, gives the dimension of displacement, i.e., [length].

2.4. *Source and receiver Green's functions*

In the integral representation for the scattered field, we need the Green's functions originating both at the source and the receiver points. Further, the gradient of total travel times from the source to a scattering point to the receiver are required in preparation of the GRT inversion. We introduce these functions here.

Set

$$\tilde{G}(\boldsymbol{x},t) = G(\boldsymbol{x},\boldsymbol{s},t)\ ,\quad \hat{G}(\boldsymbol{x},t) = G(\boldsymbol{r},\boldsymbol{x},t)\ . \tag{15}$$

Employing asymptotic ray theory in both Green's functions, we introduce the notation

$$\tilde{A}^{(\tilde{N})}(\boldsymbol{x}) = A^{(N)}(\boldsymbol{x},\boldsymbol{s})\ ,\quad \hat{A}^{(\hat{M})}(\boldsymbol{x}) = A^{(M)}(\boldsymbol{r},\boldsymbol{x}) \tag{16}$$

in the case of scattering from incident mode N to outgoing mode M.

According to Eq.(7), the slowness vectors at $\boldsymbol{x}$ are given by

$$\tilde{\gamma}^{(\tilde{N})}(\boldsymbol{x}) = \nabla_{\boldsymbol{x}}\tau^{(N)}(\boldsymbol{x},\boldsymbol{s})\ ,\quad \hat{\gamma}^{(\hat{M})}(\boldsymbol{x}) = \nabla_{\boldsymbol{x}}\tau^{(M)}(\boldsymbol{r},\boldsymbol{x})\ ; \tag{17}$$

the associated phase directions are given by

$$\tilde{\boldsymbol{\alpha}}^{(\tilde{N})} = \frac{\tilde{\gamma}^{(\tilde{N})}}{|\tilde{\gamma}^{(\tilde{N})}|}\ ,\quad \hat{\boldsymbol{\alpha}}^{(\hat{M})} = \frac{\hat{\gamma}^{(\hat{M})}}{|\hat{\gamma}^{(\hat{M})}|} \tag{18}$$

and the phase speeds (cf. Eq.(12)) are given by

$$\tilde{V}^{(\tilde{N})} = \frac{1}{|\tilde{\gamma}^{(\tilde{N})}|}\ ,\quad \hat{V}^{(\hat{M})} = \frac{1}{|\hat{\gamma}^{(\hat{M})}|}\ . \tag{19}$$

We also define the two-way travel time $T^{(\tilde{N}\hat{M})}$ and its gradient,

$$\begin{aligned} T^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{y},\boldsymbol{s}) &\equiv \tau^{(N)}(\boldsymbol{y},\boldsymbol{s}) + \tau^{(M)}(\boldsymbol{r},\boldsymbol{y}) \quad , \\ \boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s}) &\equiv \nabla_{\boldsymbol{x}} T^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s}) \quad . \end{aligned} \tag{20}$$

From Eq.(17) we see that

$$\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s}) = \tilde{\gamma}^{(\tilde{N})}(\boldsymbol{x}) + \hat{\gamma}^{(\hat{M})}(\boldsymbol{x})\ . \tag{21}$$

The direction of $\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}$,

$$\boldsymbol{\nu} \equiv \frac{\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}}{|\boldsymbol{\Gamma}^{(\tilde{N}\hat{M})}|}\ ,$$

will be the *migration dip*, which we referred to in the introduction. The ray geometry and slowness vectors are illustrated in figure 1; the associated polarization vectors are shown in figure 2.

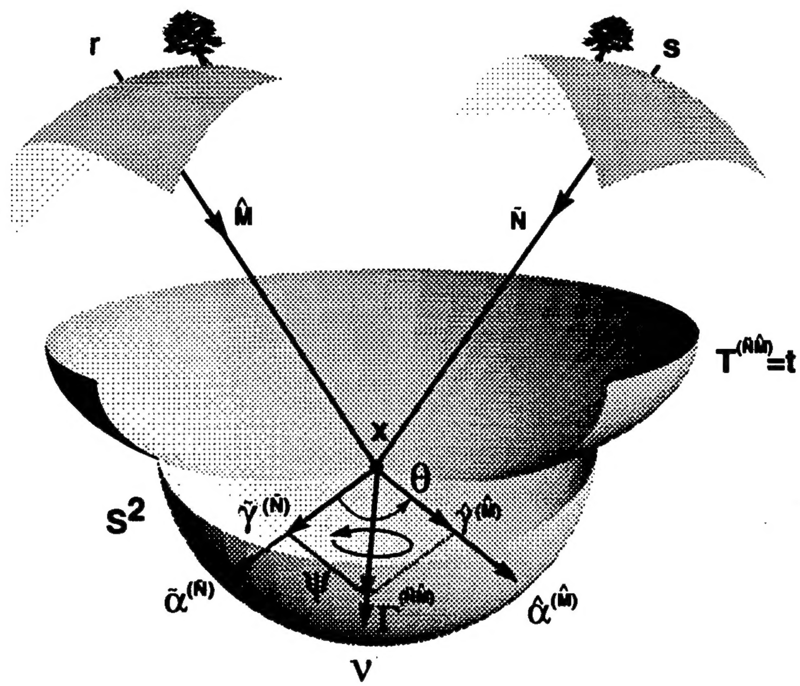

Figure 1. Source-receiver ray geometry (S^2 = unit sphere).

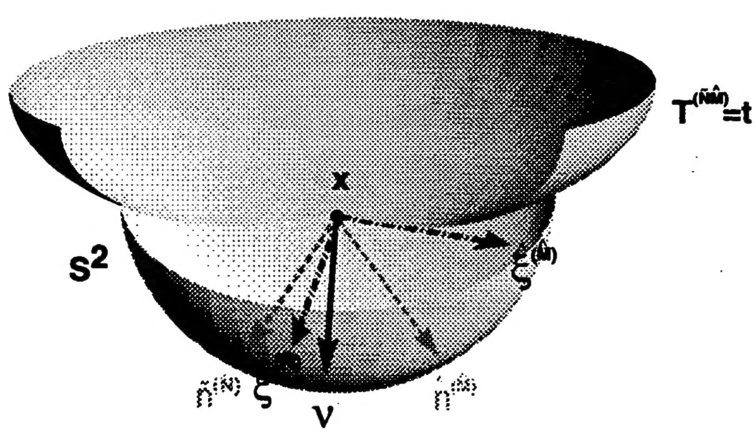

Figure 2. Source-receiver ray polarizations.

3. Medium description: micro-local perturbation

Let $\mathbf{c}(\boldsymbol{x})$ denote a smoothly varying background medium. To analyse a typical geological setting, we consider the following, micro-local, representation,

$$\mathbf{c}^{(1)}(\boldsymbol{x}) \to \mathbf{c}^{(1)}(\boldsymbol{x}, \phi(\boldsymbol{x})) \,, \tag{22}$$

of the medium's perturbation. Here, ϕ is a smooth function of $\boldsymbol{x}$, some (curved) level surfaces of which describe a family of interfaces. The gradient of the perturbation is assumed to vary rapidly normal to the level surfaces of ϕ and smoothly along them, implying that

$$\nabla_{\boldsymbol{x}} \mathbf{c}^{(1)} = (\mathbf{c}^{(1)})' \, (\nabla_{\boldsymbol{x}} \phi) + \text{ terms smoother in } \boldsymbol{x} \,, \quad (\mathbf{c}^{(1)})' = \partial_\phi \mathbf{c}^{(1)} \,. \tag{23}$$

This representation extends to a small ball around any point, in particular the image point $\boldsymbol{y}$, say, under investigation. The derivative $(\mathbf{c}^{(1)})'$ is understood in the distributional sense. Typically, it will have a Dirac-distribution type behavior across any geological interface. Loosely, the derivative can be interpreted as the difference in medium properties across a level surface. The support of $\mathbf{c}^{(1)}$ is denoted by $\mathcal{D}$.

In the background medium we assume that the ray theory of the previous section applies. In the forward problem the background $\mathbf{c}$ and the perturbation $\mathbf{c}^{(1)}$ are both known; in the inverse problem the aim is to reconstruct $\mathbf{c}^{(1)}$ given $\mathbf{c}$.

We will omit the first argument of $\mathbf{c}^{(1)}$ in the remainder of this paper, and take only the most rapidly varying term of the perturbation's derivatives into account. Imaging reflectors or interfaces amounts to mapping this leading-order behavior, $(\mathbf{c}^{(1)})'$. We will confirm below that our inversion procedure does this, and also provides estimates of angularly dependent reflection coefficients.

4. The single scattering equation

In this section, we introduce the Born approximation representing the singly scattered wavefield. We then show how to recast this volume integral representation into a surface integral for the response to the most singular element of the scattering process, namely, reflecting the discontinuities of the medium parameters.

4.1. Volume integral representation

We begin the analysis with the volume-scattering representation of the ray-Born approximation for the scattered displacement field $\boldsymbol{u}^{(1)}$ for the (NM) conversion. Let $(\boldsymbol{r}, \boldsymbol{s}) \in \partial R \times \partial S$; ideally, the boundaries $\partial R, \partial S \sim S^2$ (S^2 = unit sphere) are closed surfaces surrounding the heterogeneous domain $\mathcal{D}$. Then (De Hoop *et al.* [14])

$$u^{(1)}_{pq}(\boldsymbol{r}, \boldsymbol{s}, t) = -\int_{\mathcal{D}} \hat{\xi}^{(\hat{M})}_p(\boldsymbol{r}) \tilde{\xi}^{(\tilde{N})}_q(\boldsymbol{s}) A^{(\tilde{N}\hat{M})}(\boldsymbol{x}) \; \times \tag{24}$$
$$(\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{x}, \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x})))^T \, \mathbf{c}^{(1)}(\boldsymbol{x}) \, \delta''(t - T^{(\tilde{N}\hat{M})}(\boldsymbol{r}, \boldsymbol{x}, \boldsymbol{s})) \, \mathrm{d}\boldsymbol{x} \,,$$

where $N, M \in \{1,2,3\}$,

$$A^{(\tilde{N}\hat{M})}(\boldsymbol{x}) = \rho(\boldsymbol{x})\, \tilde{A}^{(\tilde{N})}(\boldsymbol{x})\hat{A}^{(\hat{M})}(\boldsymbol{x})\,, \tag{25}$$

contains the amplitudes,

$$\begin{aligned}
\mathbf{w}^{(\tilde{N}\hat{M})} &= \left\{\tilde{\xi}_i^{(\tilde{N})}\hat{\xi}_i^{(\hat{M})}, \tfrac{1}{2}\left[\hat{a}_{ij}^{(\hat{M})}\tilde{a}_{k\ell}^{(\tilde{N})} + \hat{a}_{k\ell}^{(\hat{M})}\tilde{a}_{ij}^{(\tilde{N})}\right]\right\}\,,\\
\hat{a}_{ij}^{(\hat{M})} &= \tfrac{1}{2}V_\circ^{(\hat{M})}\,(\hat{\xi}_i^{(\hat{M})}\hat{\gamma}_j^{(\hat{M})} + \hat{\xi}_j^{(\hat{M})}\hat{\gamma}_i^{(\hat{M})})\,,\\
\tilde{a}_{k\ell}^{(\tilde{N})} &= \tfrac{1}{2}V_\circ^{(\tilde{N})}\,(\tilde{\xi}_k^{(\tilde{N})}\tilde{\gamma}_\ell^{(\tilde{N})} + \tilde{\xi}_\ell^{(\tilde{N})}\tilde{\gamma}_k^{(\tilde{N})})\,,
\end{aligned}$$

describes the contrast-source radiation patterns, and

$$\mathbf{c}^{(1)} = \left\{\frac{\rho^{(1)}}{\rho}, \frac{c_{ijk\ell}^{(1)}}{\rho\, V_\circ^{(\hat{M})}V_\circ^{(\tilde{N})}}\right\}\,,$$

represents the relative medium perturbation. Here, $V_\circ^{(L)}$ denotes the (local) normal speed of mode L in the background medium averaged over all phase directions. By introducing this convenient scale, we have made the quantities $\hat{a}_{ij}^{(\hat{M})}$, $\tilde{a}_{k\ell}^{(\tilde{N})}$, and $\mathbf{c}^{(1)}$ dimensionless since the γ's have the dimensions of [slowness] and c_{ijkl} has the dimension of [density]$\times$[velocity]^2. The notation $_\circ$ is meant to emphasize that the quantity is angle independent, which is important for retaining the actual medium perturbation from $\mathbf{c}^{(1)}$ and the GRT inversion to be applicable.

In the micro-local setting, substituting Eq.(22) into Eq.(24), we have

$$\begin{aligned}
u_{pq}^{(1)}(\boldsymbol{r},\boldsymbol{s},t) = -\int_{\mathcal{D}} &\hat{\xi}_p^{(\hat{M})}(\boldsymbol{r})\tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s})A^{(\tilde{N}\hat{M})}(\boldsymbol{x})\,\times\\
&(\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{x},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x})))^T\,\mathbf{c}^{(1)}(\phi(\boldsymbol{x}))\,\delta''(t-T(\boldsymbol{x}))\;\mathrm{d}\boldsymbol{x}\,,
\end{aligned} \tag{26}$$

where, for convenience, we employ the shorthand notation

$$T(\boldsymbol{x}) = T^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s})\,,\quad \boldsymbol{\Gamma}(\boldsymbol{x}) = \nabla_{\boldsymbol{x}}T^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s})\,, \tag{27}$$

see Eq.(20). To make use of the properties of the gradient of the medium's perturbation (cf. Eq.(23)), we will partially integrate expression (26). Since

$$(\nabla_{\boldsymbol{x}}T)(\boldsymbol{x})\,\delta''(t-T(\boldsymbol{x})) = -\nabla_{\boldsymbol{x}}\delta'(t-T(\boldsymbol{x}))\,, \tag{28}$$

we have,

$$\delta''(t-T(\boldsymbol{x})) = -\frac{1}{(\bar{\boldsymbol{\nu}}\cdot\nabla_{\boldsymbol{x}}T)(\boldsymbol{x})}\,\bar{\boldsymbol{\nu}}\cdot\nabla_{\boldsymbol{x}}\delta'(t-T(\boldsymbol{x}))\,, \tag{29}$$

for arbitrary $\bar{\boldsymbol{\nu}} \in S^2$ as long as $\bar{\boldsymbol{\nu}}\cdot\nabla_{\boldsymbol{x}}T \neq 0$. Hence,

$$\begin{aligned}
u_{pq}^{(1)}(\boldsymbol{r},\boldsymbol{s},t) \simeq -\int_{\mathcal{D}} &\hat{\xi}_p^{(\hat{M})}(\boldsymbol{r})\tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s})A^{(\tilde{N}\hat{M})}(\boldsymbol{x})\,\times\\
&(\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{x},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x})))^T\,(\mathbf{c}^{(1)})'(\phi(\boldsymbol{x}))\,\times\\
&\left.\frac{(\bar{\boldsymbol{\nu}}\cdot\nabla_{\boldsymbol{x}}\phi)}{(\bar{\boldsymbol{\nu}}\cdot\boldsymbol{\Gamma})}\right|_{\boldsymbol{x}}\delta'(t-T(\boldsymbol{x}))\;\mathrm{d}\boldsymbol{x}\,.
\end{aligned} \tag{30}$$

Here, the approximation arises from neglecting lower order terms as in Eq.(23), as well as neglecting derivatives of the remaining amplitude in Eq.(26), compared to $(\mathbf{c}^{(1)})'$. These will produce asymptotically lower order contributions to the wavefield.

It is assumed that $\bar{\boldsymbol{\nu}} = \bar{\boldsymbol{\nu}}(\boldsymbol{x})$ is slowly varying in space, and may be chosen equal to the local *geological dip*,

$$\boldsymbol{\nu}_\phi \equiv \frac{\nabla_{\boldsymbol{x}}\phi}{|\nabla_{\boldsymbol{x}}\phi|} \,. \tag{31}$$

On the other hand, we can choose $\bar{\boldsymbol{\nu}} = \boldsymbol{\nu}$, which we always know. In the inverse scattering problem, the geological dip is unknown and has to be determined. Below, we show that at stationarity the geological and migration dips must be parallel.

4.2. *Surface integral representation*

Now, we show how to recast the volume integral representation (30) into an integral over surface integrals over the level surfaces of ϕ. To this end, we choose curvi-linear coordinates $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{x})$, $\boldsymbol{\sigma} = (\sigma_\mu, \sigma_3)$, $\mu = 1, 2$, such that the σ_μ are coordinates in the level surfaces of ϕ and σ_3 is the local coordinate in the $\boldsymbol{\nu}_\phi$-direction. If σ_3 represents the actual value of the level of ϕ, we set $\sigma_3 = L$. The volume form is given by

$$\mathrm{d}\boldsymbol{x} = \frac{1}{|\nabla_{\boldsymbol{x}}\phi|}\,\mathrm{d}L\;\mathrm{d}\Sigma(\boldsymbol{x})\;,\quad \mathrm{d}\Sigma(\boldsymbol{x}) = |\partial_{\sigma_1}\boldsymbol{x} \wedge \partial_{\sigma_2}\boldsymbol{x}|\;\mathrm{d}\sigma_1\mathrm{d}\sigma_2\;;$$

the transformation from Cartesian to curvi-linear coordinates yields the Jacobian

$$\frac{\partial(\boldsymbol{x})}{\partial(\boldsymbol{\sigma})} = |\partial_{\sigma_3}\boldsymbol{x}\cdot(\partial_{\sigma_1}\boldsymbol{x} \wedge \partial_{\sigma_2}\boldsymbol{x})| = \frac{|\partial_{\sigma_1}\boldsymbol{x} \wedge \partial_{\sigma_2}\boldsymbol{x}|}{|\nabla_{\boldsymbol{x}}\phi|}\,. \tag{32}$$

Now write

$$(\mathbf{c}^{(1)})'(\phi(\boldsymbol{x})) = \int_{\mathbf{R}} (\mathbf{c}^{(1)})'(L)\,\delta(\phi(\boldsymbol{x}) - L)\;\mathrm{d}L\,. \tag{33}$$

Substituting Eq.(33) into Eq.(30), interchanging the order of integration, and using the property (cf. Eq.(32))

$$\int_D \cdots \delta(\phi(\boldsymbol{x}) - L)\,\mathrm{d}\boldsymbol{x} = \int_{\phi=L} \cdots \frac{1}{|\nabla_{\boldsymbol{x}}\phi|}\,\mathrm{d}\Sigma(\boldsymbol{x})\,,$$

we obtain

$$\begin{aligned} u^{(1)}_{pq}(\boldsymbol{r},\boldsymbol{s},t) \simeq -\int_{\mathbf{R}} \mathrm{d}L \Bigg[\int_{\phi=L} & \hat{\xi}^{(\hat{M})}_p(\boldsymbol{r})\tilde{\xi}^{(\tilde{N})}_q(\boldsymbol{s}) A^{(\tilde{N}\hat{M})}(\boldsymbol{x}) \;\times \\ & (\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{x}, \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x})))^T\,(\mathbf{c}^{(1)})'(L) \;\times \\ & \left.\frac{(\bar{\boldsymbol{\nu}}\cdot\nabla_{\boldsymbol{x}}\phi)}{(\bar{\boldsymbol{\nu}}\cdot\boldsymbol{\Gamma})\,|\nabla_{\boldsymbol{x}}\phi|}\right|_{\boldsymbol{x}} \delta'(t - T(\boldsymbol{x}))\,\mathrm{d}\Sigma(\boldsymbol{x})\Bigg] \\ = -\int_{\mathbf{R}} \mathrm{d}L \Bigg[\int_{\phi=L} & \hat{\xi}^{(\hat{M})}_p(\boldsymbol{r})\tilde{\xi}^{(\tilde{N})}_q(\boldsymbol{s}) A^{(\tilde{N}\hat{M})}(\boldsymbol{x}) \;\times \end{aligned}$$

$$(\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{x},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x})))^T\,(\mathbf{c}^{(1)})'(L)\;\times$$
$$\left.\frac{(\bar{\boldsymbol{\nu}}\cdot\boldsymbol{\nu}_\phi)}{(\bar{\boldsymbol{\nu}}\cdot\boldsymbol{\Gamma})}\right|_{\boldsymbol{x}}\delta'(t-T(\boldsymbol{x}))\,\mathrm{d}\Sigma(\boldsymbol{x})\Big]\;. \quad (34)$$

As a consequence of Eq.(34), we also have

$$\begin{aligned}\partial_t u^{(1)}_{pq}(\boldsymbol{r},\boldsymbol{s},t) = -\int_{\mathbf{R}}\mathrm{d}L\,\Big[\iint_{\phi=L}\hat{\xi}^{(\hat{M})}_p(\boldsymbol{r})\tilde{\xi}^{(\tilde{N})}_q(\boldsymbol{s})A^{(\tilde{N}\hat{M})}(\boldsymbol{x})\;\times\\ (\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{x},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x})))^T\,(\mathbf{c}^{(1)})'(L)\;\times\\ \left.\frac{(\bar{\boldsymbol{\nu}}\cdot\boldsymbol{\nu}_\phi)}{(\bar{\boldsymbol{\nu}}\cdot\boldsymbol{\Gamma})}\right|_{\boldsymbol{x}}\delta''(t-T(\boldsymbol{x}))\,\mathrm{d}\Sigma(\boldsymbol{x})\Big]\;.\end{aligned} \quad (35)$$

The singular support of the medium perturbation, $\phi(\boldsymbol{x}) = L$, and the isochrone surfaces, $T(\boldsymbol{x}) = t$, are illustrated in figure 3.

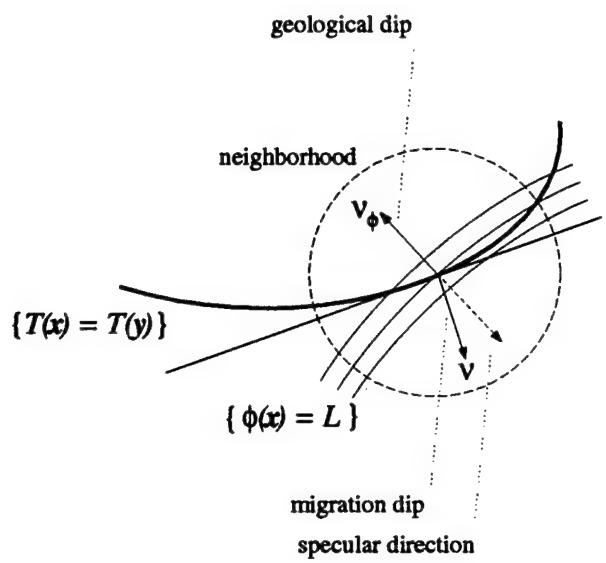

Figure 3. Micro-local medium perturbation.

5. Reflection and transmission coefficients

To identify reflection and transmission coefficients in Eq.(35), we first extract phase velocities at the scattering point from the amplitudes,

$$A_u(\boldsymbol{x}) = A^{(\tilde{N}\hat{M})}(\boldsymbol{x})\,\left[\tilde{V}^{(\tilde{N})}(\boldsymbol{x})(\hat{V}^{(\hat{M})}(\boldsymbol{x}))^3\right]^{1/2}\;; \quad (36)$$

then, Eq.(35) can be written in the form

$$\partial_t u_{pq}^{(1)}(\boldsymbol{r},\boldsymbol{s},t) = -\int_{\mathbf{R}} \Big[\int_{\phi=L} \hat{\xi}_p^{(\hat{M})}(\boldsymbol{r}) \tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s}) A_u(\boldsymbol{x}) \times \tag{37}$$
$$R_L^{(\tilde{N}\hat{M})}(\boldsymbol{x}, \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x}))\, (\bar{\boldsymbol{\nu}}\cdot\boldsymbol{\nu}_\phi)(\bar{\boldsymbol{\nu}}\cdot\boldsymbol{\Gamma})\,|_{\boldsymbol{x}}\ \delta''(t - T(\boldsymbol{x}))\, \frac{\mathrm{d}\Sigma(\boldsymbol{x})}{|\nabla_{\boldsymbol{x}}\phi|} \Big]\, \mathrm{d}L\,,$$

in which,

$$R_L^{(\tilde{N}\hat{M})}(.,\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(.)) = \frac{(\mathbf{w}^{(\tilde{N}\hat{M})})^T\, (\mathbf{c}^{(1)})' |\nabla_{\boldsymbol{x}}\phi|}{\left[\tilde{V}^{(\tilde{N})}(\hat{V}^{(\hat{M})})^3\right]^{1/2} |\boldsymbol{\Gamma}|^2\, (\bar{\boldsymbol{\nu}}\cdot\boldsymbol{\nu})^2}\,, \tag{38}$$

represents the scattering coefficient for the (N, M) conversion at $\boldsymbol{x}$ with $\phi(\boldsymbol{x}) = L$, i.e., $R_L^{(\tilde{N}\hat{M})}$ really depends on $\sigma_\mu(\boldsymbol{x})$, $\mu = 1, 2$. Observe that the scattering coefficient contains a function that may be singular on the level surfaces of ϕ. In fact, the integration over L picks up the singular support of $(\mathbf{c}^{(1)})'$.

5.1. *Specular reflection and transmission*

Now, let $\mathbf{c}^{(1)}$ contain a step function in L (across a curved interface at $L = L_0$). Then $(\mathbf{c}^{(1)})'(L) = (\Delta\mathbf{c}^{(1)})\,\delta(L - L_0)$. At the specular point for given $\boldsymbol{\nu}_\phi$, the pair $\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(.)$ satisfies Snell's law,

$$\frac{\hat{\boldsymbol{\alpha}}^{(\hat{M})}}{\hat{V}^{(\hat{M})}}\cdot(\mathbf{I} - \boldsymbol{\nu}_\phi\boldsymbol{\nu}_\phi) = -\frac{\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}}{\tilde{V}^{(\tilde{N})}}\cdot(\mathbf{I} - \boldsymbol{\nu}_\phi\boldsymbol{\nu}_\phi)\ . \tag{39}$$

[In an isotropic medium with $M = N$ the solution is simply given by

$$\hat{\boldsymbol{\alpha}}_s^{(\hat{N})} = -\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}\cdot(\mathbf{I} - 2\boldsymbol{\nu}_\phi\boldsymbol{\nu}_\phi)\ ,$$

representing ordinary reflection.] At the specular point, the migration dip coincides with the geological dip, $\boldsymbol{\nu} = \boldsymbol{\nu}_\phi$.

Substituting the solution, $\hat{\boldsymbol{\alpha}}_s^{(\hat{M})}$, of Eq.(39) into the scattering matrix Eq.(38), yields the *linearised reflection/transmission coefficients* $\mathsf{r}^{(\tilde{N}\hat{M})}$,

$$\mathsf{r}_{L_0}^{(\tilde{N}\hat{M})}(.,\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.))\,\delta(L - L_0) = R_L^{(\tilde{N}\hat{M})}(.,\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.), \hat{\boldsymbol{\alpha}}_s^{(\hat{M})}(.))\ . \tag{40}$$

Here, the left side exploits the evaluation of $\hat{\boldsymbol{\alpha}}^{(\hat{M})}$ at specular and explicitly notes the distributional character of the right side. In fact, with $\bar{\boldsymbol{\nu}} = \boldsymbol{\nu}_\phi$, Eq.(37) is equivalent to the Kirchhoff-Born approximation, which satisfies the principle of reciprocity. For notational convenience, we introduce

$$\mathsf{R}_L^{(\tilde{N}\hat{M})}(.,\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.)) = \mathsf{r}_{L_0}^{(\tilde{N}\hat{M})}(.,\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.))\,\delta(L - L_0) \tag{41}$$

to absorb the Dirac distribution in $\mathsf{R}_L^{(\tilde{N}\hat{M})}$.

5.2. The Kirchhoff approximation

Here, we show how to go from the linearised Born representation Eq.(35) to the non-linear Kirchhoff approximation. For general reference, we introduce the relevant scattering and specular angles: in addition to $\boldsymbol{\nu}$, we set

$$\cos\theta = \tilde{\boldsymbol{\alpha}}^{(\tilde{N})} \cdot \hat{\boldsymbol{\alpha}}^{(\hat{M})}, \quad \psi = \text{third Euler angle} . \tag{42}$$

Then, at any point in $\mathcal{D}$, we have a mapping

$$(\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(.)) \rightarrow (\boldsymbol{\nu}_., \theta_., \psi_.) . \tag{43}$$

If $\boldsymbol{\nu} = \boldsymbol{\nu}_\phi$, the ray geometry is specular; let the associated specular scattering angles be defined as

$$\cos\tilde{\theta} = \tilde{\alpha}^{(\tilde{N})} \cdot \boldsymbol{\nu}_\phi, \quad \cos\hat{\theta} = \hat{\alpha}_{\mathbf{s}}^{(\hat{M})} \cdot \boldsymbol{\nu}_\phi, \quad \theta_{\mathbf{s}} = \tilde{\theta} - \hat{\theta} . \tag{44}$$

In the non-reciprocal, 'non-linear' Kirchhoff approximation, the scattering matrix is simply replaced by the *full* reflection/transmission coefficients at specular, cf. Eq.(37) with $\bar{\boldsymbol{\nu}} = \boldsymbol{\nu}_\phi$,

$$\partial_t u_{pq}^{(1)}(\boldsymbol{r}, \boldsymbol{s}, t) = -\int_{\mathbf{R}} \left[\iint_{\phi=L} \hat{\xi}_p^{(\hat{M})}(\boldsymbol{r}) \tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s}) A_u(\boldsymbol{x}) \times \right. \tag{45}$$
$$\left. \mathsf{R}_L^{(\tilde{N}\hat{M})}(\boldsymbol{x}, \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}))\, (\boldsymbol{\nu}_\phi \cdot \boldsymbol{\Gamma})\,|_{\boldsymbol{x}}\, \delta''(t - T(\boldsymbol{x}))\, \frac{\mathrm{d}\Sigma(\boldsymbol{x})}{|\nabla_{\boldsymbol{x}} \phi|} \right] \mathrm{d}L .$$

For wide-angle (θ) scattering, this representation is certainly more adequate than Eq.(37). The time derivative is taken to pave the way for the Radon transform inversion. (In three dimensions, one needs the second derivative of the Dirac distribution.) In our further analysis, we actually employ the reciprocal representation Eq.(37) in which $R_L^{(\tilde{N}\hat{M})}$ is replaced by the full reflection/transmission coefficient at specular. Note that due to the singular function contained in $\mathsf{R}_L^{(\tilde{N}\hat{M})}$, cf. Eq.(40), the integration over L in Eq.(45) reduces to a sum over scattering surfaces or interfaces.

6. Stationary phase analysis of the direct scattering problem

By applying stationary phase arguments, the integral in Eq.(37) can be evaluated. The analysis confirms the consistency with asymptotic ray theory in configurations with a family of surface scatterers.

We choose rotated Cartesian coordinates (x_μ, z), $\mu = 1, 2$ in the neighborhood of a yet-to-be determined *specular point* $\boldsymbol{y}(L) \in \{\phi = L\}$, such that

$$z \parallel \boldsymbol{\nu}_\phi, \quad \{x_\mu\} \perp \boldsymbol{\nu}_\phi \tag{46}$$

and

$$\boldsymbol{\nu} = \boldsymbol{\nu}_\phi ;$$

see figure 3. The function $T(\boldsymbol{y}(L))$ has a derivative given by

$$\frac{\mathrm{d}T}{\mathrm{d}L}=\left.\frac{|\nabla_{\boldsymbol{x}}T|}{|\nabla_{\boldsymbol{x}}\phi|}\right|_{\boldsymbol{y}(L)}, \tag{47}$$

while, by the implicit function theorem, the function $L_{\boldsymbol{y}}(t)$ satisfying

$$T\left(\boldsymbol{y}(L_{\boldsymbol{y}}(t))\right)=t$$

exists.

Taylor expansions of the level and isochrone surfaces including the curvature terms yield

$$\begin{aligned} 0=\phi(\boldsymbol{x})-\phi(\boldsymbol{y}) &= |\nabla_{\boldsymbol{x}}\phi(\boldsymbol{y})|\,z+\tfrac{1}{2}x_\mu\phi_{\mu\nu}(\boldsymbol{y})x_\nu \quad , \\ T(\boldsymbol{x}) &= T(\boldsymbol{y})+|\nabla_{\boldsymbol{x}}T(\boldsymbol{y})|\,z+\tfrac{1}{2}x_\mu T_{\mu\nu}(\boldsymbol{y})x_\nu \quad . \end{aligned} \tag{48}$$

(Summations are carried out over μ,ν.) Here,

$$\begin{aligned} \phi_{\mu\nu}(\boldsymbol{y}) &= \left.\frac{\partial^2\phi}{\partial_{x_\mu}\partial_{x_\nu}}\right|_{\boldsymbol{y}} \quad ; \\ T_{\mu\nu}(\boldsymbol{y}) &= \left.\frac{\partial^2 T}{\partial_{x_\mu}\partial_{x_\nu}}\right|_{\boldsymbol{y}} \quad . \end{aligned} \tag{49}$$

The first equality in (48) amounts to the representation of the level surface $\{\phi=L\}$,

$$z=-\frac{x_\mu\phi_{\mu\nu}x_\nu}{2\,|\nabla_{\boldsymbol{x}}\phi|},$$

which upon substitution in the second equality yields

$$T(\boldsymbol{x})=T(\boldsymbol{y})+\tfrac{1}{2}|\nabla_{\boldsymbol{x}}T(\boldsymbol{y})|\,x_\mu\Upsilon_{\mu\nu}(\boldsymbol{y})x_\nu \tag{50}$$

with

$$\Upsilon_{\mu\nu}\equiv\frac{T_{\mu\nu}}{|\nabla_{\boldsymbol{x}}T|}-\frac{\phi_{\mu\nu}}{|\nabla_{\boldsymbol{x}}\phi|}$$

and $\boldsymbol{x},\boldsymbol{y}$ in the same level surface. Note that the matrix Υ may vary with the level L, and can be negative or positive definite, or indefinite. The case of vanishing Υ, leading to a caustic analysis, will be postponed to a future paper. Otherwise, for t near $T(\boldsymbol{y})$, we have the intermediate result

$$\begin{aligned} \int_{\phi=L}\delta'(t-T(\boldsymbol{x}))\,\mathrm{d}\Sigma(\boldsymbol{x}) &= \partial_t^2\int_{\phi=L}H(t-T(\boldsymbol{x}))\,\mathrm{d}\Sigma(\boldsymbol{x}) \\ &\simeq \partial_t^2\int_{\mathbf{R}^2}H\left(t-T(\boldsymbol{y})-\tfrac{1}{2}|\nabla_{\boldsymbol{x}}T(\boldsymbol{y})|\,x_\mu\Upsilon_{\mu\nu}(\boldsymbol{y})x_\nu\right)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &= \frac{2\pi\,\delta^*(t-T(\boldsymbol{y}))}{|\boldsymbol{\Gamma}(\boldsymbol{y})|\sqrt{|\det(\Upsilon(\boldsymbol{y}))|}}, \end{aligned} \tag{51}$$

using tangent plane coordinates, and where

$$\delta^* = \begin{cases} \delta & \text{if } \Upsilon \text{ positive} \\ \mathcal{H}\delta & \text{if } \Upsilon \text{ indefinite} \\ -\delta & \text{if } \Upsilon \text{ negative}. \end{cases} \tag{52}$$

In the above $\mathcal{H}$ denotes the Hilbert transform; the notation * indicates an action on the time dependence.

We will use Eq.(51) to evaluate the integral in Eq.(34) eventually. First, note that Eq.(51) implies

$$\begin{aligned} &\int_{\mathbf{R}} \mathrm{d}L \left[\int_{\phi=L} (\mathbf{c}^{(1)})'(L) \left.\frac{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\nu}_\phi)}{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\Gamma})}\right|_{\boldsymbol{x}} \delta'(t - T(\boldsymbol{x}))\, \mathrm{d}\Sigma(\boldsymbol{x}) \right] \\ &\quad \simeq \int_{\mathbf{R}} \mathrm{d}L\, (\mathbf{c}^{(1)})'(L) \frac{2\pi}{|\boldsymbol{\Gamma}|\sqrt{|\det(\Upsilon)|}} \left.\frac{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\nu}_\phi)}{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\Gamma})}\right|_{\boldsymbol{y}(L)} \delta^* [t - T(\boldsymbol{y}(L))] \\ &\quad = \frac{2\pi}{|\boldsymbol{\Gamma}|\sqrt{|\det(\Upsilon)|}} \left.\frac{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\nu}_\phi)}{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\Gamma})}\right|_{\boldsymbol{y}(L_{\boldsymbol{y}}(t))} \left(\frac{\mathrm{d}T}{\mathrm{d}L}\right)^{-1} (\mathbf{c}^{(1)})' \Big|^*_{L_{\boldsymbol{y}}(t)}, \end{aligned} \tag{53}$$

where, for any function f,

$$\left(\frac{\mathrm{d}T}{\mathrm{d}L}\right)^{-1} \mathsf{f} \Big|^*_{L_{\boldsymbol{y}}(t)} = \int_{\mathbf{R}} \mathsf{f}(L)\, \delta^*[t - T(\boldsymbol{y}(L))]\, \mathrm{d}L\,.$$

Substituting Eq.(47) into Eq.(53), and using the result in Eq.(34) implies

$$\begin{aligned} u^{(1)}_{pq}(\boldsymbol{r}, \boldsymbol{s}, t) \simeq &-\frac{2\pi\, |\nabla_{\boldsymbol{x}} \phi|}{\sqrt{|\det(\Upsilon)|}\, |\boldsymbol{\Gamma}|^3} \hat{\xi}^{(\hat{M})}_p(\boldsymbol{r}) \tilde{\xi}^{(\tilde{N})}_q(\boldsymbol{s}) A^{(\tilde{N}\hat{M})}(.) \times \\ &(\mathbf{w}^{(\tilde{N}\hat{M})}(., \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.), \hat{\boldsymbol{\alpha}}^{(\hat{M})}_s(.)))^T \Big|_{\boldsymbol{y}(L_{\boldsymbol{y}}(t))} (\mathbf{c}^{(1)})' \Big|^*_{L_{\boldsymbol{y}}(t)}, \end{aligned} \tag{54}$$

since at the specular point we have $\boldsymbol{\nu} = \boldsymbol{\nu}_\phi$ and $\boldsymbol{\nu}_\phi \cdot \boldsymbol{\Gamma} = |\boldsymbol{\Gamma}|$. This formula is an extension of the *convolutional model* approximation in one-dimensional space to three dimensions.

In terms of the reflection/transmission coefficients, we have

$$u^{(1)}_{pq}(\boldsymbol{r}, \boldsymbol{s}, t) \simeq -\frac{2\pi}{\sqrt{|\det(\Upsilon)|}\, |\boldsymbol{\Gamma}|} \hat{\xi}^{(\hat{M})}_p(\boldsymbol{r}) \tilde{\xi}^{(\tilde{N})}_q(\boldsymbol{s}) A_u(.)\, \mathsf{R}^{(\tilde{N}\hat{M})}_L \Big|^*_{\substack{\boldsymbol{y} = \boldsymbol{y}(L) \\ L = L_{\boldsymbol{y}}(t)}}$$

[note that the * relates to the KMAH index, see e.g. Hörmander [22]]. To verify this result, use Eqs.(36), (38), (40), and (41) in (54).

7. Inversion based on the GRT

In preparation of the inverse transformation, we introduce the scalar quantity on the diffraction surface,

$$\partial_t u^{(\tilde{N}\hat{M})}(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{y}) = \frac{\hat{\xi}_p^{(\hat{M})}(\boldsymbol{r})\, \partial_t u_{pq}^{(1)}(\boldsymbol{r}, \boldsymbol{s}, T^{(\tilde{N}\hat{M})}(\boldsymbol{r}, \boldsymbol{y}, \boldsymbol{s}))\, \tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s})}{A^{(\tilde{N}\hat{M})}(\boldsymbol{y})} . \tag{55}$$

Then, using Eq.(37), we get

$$\frac{\partial_t u^{(\tilde{N}\hat{M})}(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{y})}{\left[\tilde{V}^{(\tilde{N})}(\boldsymbol{y})(\hat{V}^{(\hat{M})}(\boldsymbol{y}))^3\right]^{1/2}} \simeq -\int_{\mathbb{R}} \left[\int_{\phi=L} R_L^{(\tilde{N}\hat{M})}(\boldsymbol{x}, \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x})) \times \right.$$

$$\left. (\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\nu}_\phi)(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\Gamma})\,|_{\boldsymbol{x}}\, \delta''(T(\boldsymbol{y}) - T(\boldsymbol{x}))\, \frac{\mathrm{d}\Sigma(\boldsymbol{x})}{|\nabla_{\boldsymbol{x}}\phi|}\right] \mathrm{d}L . \tag{56}$$

Here, we have set $A_u(\boldsymbol{x})/A_u(\boldsymbol{y}) = 1$, which is its value at the dominant critical point of the integrand. Furthermore, we will exploit the expansion

$$T(\boldsymbol{y}) - T(\boldsymbol{x}) = \boldsymbol{\Gamma}(\boldsymbol{y}) \cdot (\boldsymbol{y} - \boldsymbol{x}) + \ldots \simeq |\boldsymbol{\Gamma}(\boldsymbol{y})|\, (\boldsymbol{y} - \boldsymbol{x}) \cdot \boldsymbol{\nu} , \tag{57}$$

which describes the tangent plane to the isochron at $\boldsymbol{y}$. In our further analysis, we will make use of the identity

$$\delta''(|\boldsymbol{\Gamma}(\boldsymbol{y})|\, (\boldsymbol{y} - \boldsymbol{x}) \cdot \boldsymbol{\nu}) = |\boldsymbol{\Gamma}(\boldsymbol{y})|^{-3}\, \delta''((\boldsymbol{y} - \boldsymbol{x}) \cdot \boldsymbol{\nu}) .$$

7.1. Linearised inversion

The basis of the GRT inversion is Gel'fand's plane wave expansion, which we write in the form

$$-8\pi^2 \delta(\phi(\boldsymbol{y}) - L) = \int_{S^2} \int_D \delta(\phi(\boldsymbol{x}) - L) \delta''((\boldsymbol{y} - \boldsymbol{x}) \cdot \boldsymbol{\nu})\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{\nu}$$

$$= \int_{S^2} \int_{\phi=L} \delta''((\boldsymbol{y} - \boldsymbol{x}) \cdot \boldsymbol{\nu})\, \frac{\mathrm{d}\Sigma(\boldsymbol{x})}{|\nabla_{\boldsymbol{x}}\phi|}\, \mathrm{d}\boldsymbol{\nu} . \tag{58}$$

We will use this expansion to invert Eq.(35). The inversion is accomplished by setting up a system of 22 equations, using the contraction according to Eq.(55), and employing $\tilde{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\alpha}}$ as variables of integration, i.e.,

$$\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{y}, \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{y}), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{y}))\, \partial_t u^{(\tilde{N}\hat{M})}(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{y})\, \left.\frac{|\boldsymbol{\Gamma}|^4}{(\boldsymbol{\nu} \cdot \boldsymbol{\nu}_\phi)}\right|_{\boldsymbol{y}} \left.\frac{\partial(\tilde{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{s}, \boldsymbol{r})}\right|_{\boldsymbol{y}} \mathrm{d}\boldsymbol{s}\mathrm{d}\boldsymbol{r}$$

$$= -\int_{\mathbb{R}} \mathrm{d}L \left[\int_{\phi=L}\right.$$

$$\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{y}, \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{y}), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{y}))\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{x}, \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x}))^T (\mathbf{c}^{(1)})'(L) \times$$

$$\left.\frac{(\boldsymbol{\nu} \cdot \boldsymbol{\Gamma})}{(\boldsymbol{\nu} \cdot \boldsymbol{\nu}_\phi)}\right|_{\boldsymbol{y}} \left.\frac{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\nu}_\phi)}{(\bar{\boldsymbol{\nu}} \cdot \boldsymbol{\Gamma})}\right|_{\boldsymbol{x}} \left.\frac{\partial(\tilde{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{\nu}, \theta, \psi)}\right|_{\boldsymbol{y}} \left.\delta''((\boldsymbol{y} - \boldsymbol{x}) \cdot \boldsymbol{\nu})\, \mathrm{d}\Sigma(\boldsymbol{x})\right] \mathrm{d}\boldsymbol{\nu}\mathrm{d}\theta\mathrm{d}\psi .$$

(59)

Define the matrix

$$\Lambda_{.}(\boldsymbol{\nu}_{.}) \equiv \int_{S^2} \tag{60}$$
$$\mathbf{w}^{(\tilde{N}\hat{M})}(.,\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(.))\mathbf{w}^{(\tilde{N}\hat{M})}(.,\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(.),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(.))^T \left.\frac{\partial(\tilde{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{\nu},\theta,\psi)}\right|_{.} \mathrm{d}\theta \mathrm{d}\psi$$

at any image point $\boldsymbol{y}$. Then, employing Eq.(58) in Eq.(59) and setting $\bar{\boldsymbol{\nu}} = \boldsymbol{\nu}$ (recall that $\bar{\boldsymbol{\nu}}$ has been arbitrary until now), yields

$$(\mathbf{c}^{(1)})'(\phi(\boldsymbol{y}))\,|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{y}} = \int_{\mathbf{R}} (\mathbf{c}^{(1)})'(L)\,\delta(\phi(\boldsymbol{y}) - L)\,|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{y}}\,\mathrm{d}L$$
$$\simeq \frac{1}{8\pi^2}\int_{\partial S\times\partial R} \Lambda_{\boldsymbol{y}}^{-1}\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{y},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{y}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{y}))\,\partial_t u^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{s},\boldsymbol{y})\,\times$$
$$\left.\frac{|\Gamma|^4}{(\boldsymbol{\nu}\cdot\boldsymbol{\nu}_\phi)}\right|_{\boldsymbol{y}} \left.\frac{\partial(\tilde{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{s},\boldsymbol{r})}\right|_{\boldsymbol{y}} \mathrm{d}\boldsymbol{s}\mathrm{d}\boldsymbol{r}\;. \tag{61}$$

Note that in this inversion formula the actual shape of the level surfaces of ϕ is not used; just the local normal at the image point plays a rôle. However, in Section 9 it will be argued that the inner product, $(\boldsymbol{\nu}\cdot\boldsymbol{\nu}_\phi)$, can be removed from the formula without changing the resolution analysis in the high-frequency approximation. The inverse of Λ is understood in the *generalised* sense. We will denote the reconstruction as $\langle(\mathbf{c}^{(1)})'(\phi(\boldsymbol{y}))\,|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{y}}\rangle$.

Through Eq.(59), we have the freedom to carry out the inversion Eq.(61) in two steps. For a given image point $\boldsymbol{y}$, we can imagine the data [$(\boldsymbol{s},\boldsymbol{r})$ pairs] to be sorted into common (θ,ψ) gathers. The variable in such gathers is the migration dip $\boldsymbol{\nu}$; formally, we denote these gathers by $(\partial S\times\partial R)'(\boldsymbol{y};\theta,\psi)$. The integration over dip is then carried out prior to the integration over scattering angle and azimuth. [In practice, shooting the rays from $\boldsymbol{y}$ would be controlled by $\tilde{\boldsymbol{\alpha}}$; $\hat{\boldsymbol{\alpha}}$ then follows from (θ,ψ). The data would be simply taken at those locations where the rays intersect ∂S and ∂R, respectively.]

To control the illumination of the image point at a given dip, we introduce the partition of unity, $\{\chi_J\}$, and

$$\langle(\mathbf{c}^{(1)})'(\phi(\boldsymbol{y}))\,|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{y}}\rangle$$
$$=\sum_J \frac{1}{8\pi^2}\int_{\partial S\times\partial R} \Lambda_{\boldsymbol{y}}^{-1}\mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{y},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{y}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{y}))\,\partial_t u^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{s},\boldsymbol{y})\,\times$$
$$\left.\frac{|\Gamma|^4}{(\boldsymbol{\nu}\cdot\boldsymbol{\nu}_\phi)}\right|_{\boldsymbol{y}} \chi_J(\boldsymbol{s},\boldsymbol{r}) \left.\frac{\partial(\tilde{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{s},\boldsymbol{r})}\right|_{\boldsymbol{y}} \mathrm{d}\boldsymbol{s}\mathrm{d}\boldsymbol{r}\;. \tag{62}$$

Each term in the summation represents a partial reconstruction.

7.2. *Resolution analysis*

Backsubstituting the Kirchhoff-Born scattering formula Eq.(35) with $\bar{\boldsymbol{\nu}} = \boldsymbol{\nu}$, into the inversion formula Eq.(62) and substituting the one-sided Fourier representation of the Dirac distribution,

$$\delta''(T(\boldsymbol{y}) - T(\boldsymbol{x})) = -\mathrm{Re}\,\frac{1}{\pi}\int_{\mathbf{R}^+} \exp[i\omega\,(T(\boldsymbol{y}) - T(\boldsymbol{x}))]\,\omega^2 \mathrm{d}\omega\ ,$$

yields the *resolution* operator,

$$\begin{aligned}&\langle(\mathrm{c}^{(1)})'(\phi(\boldsymbol{y}))\,|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{y}}\rangle\\ &\quad = \int_{\mathbf{R}}\int_{\phi=L} \mathcal{R}(\boldsymbol{y},\boldsymbol{x})\,(\mathrm{c}^{(1)})'(\phi(\boldsymbol{x}))\,|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{x}}\,\frac{\mathrm{d}\Sigma(\boldsymbol{x})}{|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{x}}}\,\mathrm{d}L\ ,\end{aligned} \tag{63}$$

with matrix kernel $\mathcal{R}$,

$$\mathcal{R}(\boldsymbol{y},\boldsymbol{x}) = \mathrm{Re}\sum_J \frac{1}{8\pi^3}\int \exp[i\Phi(\boldsymbol{y},\boldsymbol{x},\Theta)]\,a(\boldsymbol{y},\boldsymbol{x},\Theta)\,\chi_J(\Theta)\mathrm{d}\Theta\ . \tag{64}$$

Here, $\Theta = (\omega, \boldsymbol{s}, \boldsymbol{r})$ and

$$\int_{\mathbf{R}^+\times\partial S\times\partial R} \cdots\,\mathrm{d}\Theta = \int_{\partial S\times\partial R}\int_{\mathbf{R}^+} \cdots\,|\boldsymbol{\Gamma}(\boldsymbol{y})|^3\omega^2\,\mathrm{d}\omega\ \mathrm{d}\boldsymbol{s}\,\mathrm{d}\boldsymbol{r}\ . \tag{65}$$

The resolution operator expresses how well the reconstruction can be accomplished within the framework of the linear theory.

The phase function Φ of the Fourier integral operator with kernel Eq.(64) is simply given by

$$\Phi(\boldsymbol{y},\boldsymbol{x},\Theta) \equiv \omega\ (T(\boldsymbol{y}) - T(\boldsymbol{x}))\ , \tag{66}$$

while the amplitude function arises as the matrix

$$\begin{aligned}a(\boldsymbol{y},\boldsymbol{x},\Theta) \equiv{}& \frac{A_u(\boldsymbol{x})}{A_u(\boldsymbol{y})}\,\hat{\xi}_p^{(\hat{M})}(\boldsymbol{r},\boldsymbol{y})\hat{\xi}_p^{(\hat{M})}(\boldsymbol{r},\boldsymbol{x})\,\tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s},\boldsymbol{y})\tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s},\boldsymbol{x})\ \times\\ &\Big[(\Lambda_{\boldsymbol{y}}(\boldsymbol{\nu}(\boldsymbol{r},\boldsymbol{y},\boldsymbol{s})))^{-1}\,\mathrm{w}^{(\tilde{N}\hat{M})}(\boldsymbol{y},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{y}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{y}))\\ &\quad(\mathrm{w}^{(\tilde{N}\hat{M})}(\boldsymbol{x},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x})))^T\Big]\ \times\\ &\left[\frac{\tilde{V}^{(\tilde{N})}(\boldsymbol{y})(\hat{V}^{(\hat{M})}(\boldsymbol{y}))^3}{\tilde{V}^{(\tilde{N})}(\boldsymbol{x})(\hat{V}^{(\hat{M})}(\boldsymbol{x}))^3}\right]^{1/2}\frac{|\boldsymbol{\Gamma}(\boldsymbol{y})|}{|\boldsymbol{\Gamma}(\boldsymbol{x})|}\,\frac{(\boldsymbol{\nu}\cdot\boldsymbol{\nu}_\phi)_{\boldsymbol{x}}}{(\boldsymbol{\nu}\cdot\boldsymbol{\nu}_\phi)_{\boldsymbol{y}}}\,\left.\frac{\partial(\tilde{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{s},\boldsymbol{r})}\right|_{\boldsymbol{y}}\ .\end{aligned} \tag{67}$$

In anticipation of introducing the scattering coefficients (cf. Eq.(38)), we introduce the one-dimensional array of functions $\mathbf{a}_R$, satisfying

$$\begin{aligned}&\mathbf{a}_R(\boldsymbol{y},\boldsymbol{x},\Theta)\ R_L^{(\tilde{N}\hat{M})}(\boldsymbol{x},\tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{x}),\hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{x}))\\ &\qquad = a(\boldsymbol{y},\boldsymbol{x},\Theta)\,(\mathrm{c}^{(1)})'(\phi(\boldsymbol{x}))\,|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{x}}\ .\end{aligned} \tag{68}$$

We will interpret Θ in the spatial Fourier domain. To this end, we carry out two coordinate transformations. First, we employ the ray-induced mapping

$$\boldsymbol{s} = \boldsymbol{s}(\tilde{N},\hat{M},\boldsymbol{\nu},\theta,\psi),\quad \boldsymbol{r} = \boldsymbol{r}(\tilde{N},\hat{M},\boldsymbol{\nu},\theta,\psi)\ \ \textit{for fixed}\ \boldsymbol{y}\ , \tag{69}$$

with

$$\int_{\partial S\times\partial R}\cdots\left.\frac{\partial(\tilde{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{s},\boldsymbol{r})}\right|_{\boldsymbol{y}}\mathrm{d}\boldsymbol{s}\,\mathrm{d}\boldsymbol{r}=\int_{S^2\times S^2}\cdots\left.\frac{\partial(\tilde{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{\nu},\theta,\psi)}\right|_{\boldsymbol{y}}\mathrm{d}\boldsymbol{\nu}\,\mathrm{d}\theta\,\mathrm{d}\psi\,.$$

Thus, at $\boldsymbol{y}$, Θ is mapped on $(\omega,\boldsymbol{\nu},\theta,\psi)$, and we set

$$a(\boldsymbol{y},\boldsymbol{x},\omega,\boldsymbol{\nu},\theta,\psi)\left.\frac{\partial(\tilde{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{s},\boldsymbol{r})}\right|_{\boldsymbol{y}}=a(\boldsymbol{y},\boldsymbol{x},\omega,\boldsymbol{s},\boldsymbol{r})\left.\frac{\partial(\tilde{\boldsymbol{\alpha}},\hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{\nu},\theta,\psi)}\right|_{\boldsymbol{y}}.$$

We identify

$$T^{(\tilde{N}\hat{M})}(\boldsymbol{r},\boldsymbol{x},\boldsymbol{s})\quad\text{with}\quad T^{(\tilde{N}\hat{M})}(\boldsymbol{x},\boldsymbol{\nu},\theta,\psi)\,.$$

Second, the frequency ω is transformed to the wavenumber k_Γ according to

$$k_\Gamma\equiv\omega\,|\boldsymbol{\Gamma}(\boldsymbol{y})|\,;\tag{70}$$

then

$$\int_{\mathbb{R}^+}\cdots|\boldsymbol{\Gamma}(\boldsymbol{y})|^3\omega^2\,\mathrm{d}\omega=\int_{\mathbb{R}^+}\cdots k_\Gamma^2\,\mathrm{d}k_\Gamma\,.\tag{71}$$

We now identify the wave vector

$$\Theta'\equiv k_\Gamma\boldsymbol{\nu}\in\mathbb{R}^3\quad\text{with}\quad\mathrm{d}\Theta'=k_\Gamma^2\,\mathrm{d}k_\Gamma\,\mathrm{d}\boldsymbol{\nu}\,,\tag{72}$$

and we will consider (θ,ψ) as parameters, i.e., $\Theta\to(\omega,\boldsymbol{\nu},\theta,\psi)\to(\Theta',\theta,\psi)$. The Jacobian of the latter transformation is written as (cf. Eq.(71))

$$\left.\frac{\partial(\Theta')}{\partial(\omega,\boldsymbol{\nu})}\right|_{\boldsymbol{y}}=h(\boldsymbol{y},\boldsymbol{\nu})\,\omega^2\,,\quad h(\boldsymbol{y},\boldsymbol{\nu})=|\boldsymbol{\Gamma}(\boldsymbol{y})|^3\,;$$

formally, also

$$h=\left|\det\begin{pmatrix}\boldsymbol{\Gamma} & \partial_{\nu_1}\boldsymbol{\Gamma} & \partial_{\nu_2}\boldsymbol{\Gamma}\end{pmatrix}\right|\,.\tag{73}$$

The inverse transformation to frequency, the so-called Stolt mapping, is given by

$$\omega(\Theta')=\frac{\Theta'\cdot\boldsymbol{\Gamma}(\boldsymbol{y})}{|\boldsymbol{\Gamma}(\boldsymbol{y})|^2}\,,\tag{74}$$

since also

$$\Theta'=\omega\,\boldsymbol{\Gamma}(\boldsymbol{y})\,.$$

We identify

$$\Phi(\boldsymbol{y},\boldsymbol{x},\Theta)\quad\text{with}\quad\Phi(\boldsymbol{y},\boldsymbol{x},\Theta',\theta,\psi)\,.$$

In the phase space with coordinates $(\boldsymbol{x},\Theta')$, Eq.(64) gives rise to the resolution equation

$$\begin{aligned}\langle(\mathbf{c}^{(1)})'(\phi(\boldsymbol{y}))\,|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{y}}\rangle=\mathrm{Re}\sum_J\int_{S^2}\frac{1}{8\pi^3}\int_{\mathbb{R}}\int_{\phi=L}\\ \int_{\mathbb{R}^+\times S^2}\exp[i\Phi(\boldsymbol{y},\boldsymbol{x},\Theta',\theta,\psi)]\,a(\boldsymbol{y},\boldsymbol{x},\Theta',\theta,\psi)\,\chi_J(\Theta',\theta,\psi)\,\mathrm{d}\Theta'\,\times\\ (\mathbf{c}^{(1)})'(\phi(\boldsymbol{x}))\,|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{x}}\,\frac{\mathrm{d}\Sigma(\boldsymbol{x})}{|\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{x}}}\,\mathrm{d}L\;\mathrm{d}\theta\,\mathrm{d}\psi\,.\end{aligned}\tag{75}$$

For $\boldsymbol{x}$ near $\boldsymbol{y}$, $\Phi = \Theta' \cdot (\boldsymbol{y} - \boldsymbol{x}) + \cdots$. The integration over Θ' represents the spatial resolution, whereas the integration over θ, ψ primarily represents the parameter resolution per migration dip. However, note that the parameter resolution couples to the spatial resolution, since Θ' in general depends on (θ, ψ).

Below, we will constrain χ_J to be a function of $k_\Gamma = |\Theta'|, \theta, \psi$ alone.

7.3. Stationary phase analysis of the resolution operator

Inside the integral over ω in Eqs.(64)-(65), we consider ω to be large. Then, we apply a four-dimensional stationary phase analysis with respect to the integrations over $(\sigma_1, \sigma_2, \boldsymbol{\nu}) \in \mathcal{S}(L) \times S^2$, cf. Eq.(75). The range $\mathcal{S}(L)$ indicates that the surfaces are contained in $\mathcal{D}$.

We choose polar coordinates on the $\boldsymbol{\nu}$-sphere,

$$\boldsymbol{\nu} = (\sin\theta^\nu \cos\psi^\nu, \sin\theta^\nu \sin\psi^\nu, \cos\theta^\nu) \; .$$

We extract $L = \sigma_3$ and $k_\Gamma = |\Theta'|$ from the set of phase space coordinates,

$$(\boldsymbol{x}, \Theta') \to (\sigma_1, \sigma_2, L, k_\Gamma, \theta^\nu, \psi^\nu) \; , \quad \eta \equiv (\sigma_1, \sigma_2, \theta^\nu, \psi^\nu) \; ,$$

resubstitute $k_\Gamma = \omega |\boldsymbol{\Gamma}(\boldsymbol{y})|$, and set

$$\Phi(\boldsymbol{y}, \boldsymbol{x}, \Theta', \theta, \psi) = \omega \, \Phi'(\boldsymbol{y}, L, \eta, k_\Gamma, \theta, \psi) \; .$$

Writing the coordinates explicitly, the resolution equation (75) takes the form

$$\begin{gathered} \langle (\mathbf{c}^{(1)})'(\phi(\boldsymbol{y})) \, |\nabla_{\boldsymbol{x}} \phi| \boldsymbol{y} \rangle = \operatorname{Re} \sum_J \int_{S^2} \frac{1}{8\pi^3} \int_{\mathbf{R}} \int_{\mathbf{R}^+} \int_{\mathcal{S}(L) \times S^2} \\ \exp[i\omega \, \Phi'(\boldsymbol{y}, L, \eta, k_\Gamma, \theta, \psi)] \, a(\boldsymbol{y}, L, \eta, k_\Gamma, \theta, \psi) \, \chi_J(k_\Gamma, \theta, \psi) \, h(\boldsymbol{y}, \boldsymbol{\nu}) \, \times \\ (\mathbf{c}^{(1)})'(\phi(\boldsymbol{x}(\boldsymbol{\sigma}))) \, |\nabla_{\boldsymbol{x}} \phi|_{\boldsymbol{x}(\boldsymbol{\sigma})} \, \frac{\mathrm{d}\eta}{|\nabla_{\boldsymbol{x}} \phi|_{\boldsymbol{x}(\boldsymbol{\sigma})}} \, \omega^2 \mathrm{d}\omega \, \mathrm{d}L \;\; \mathrm{d}\theta \, \mathrm{d}\psi \; , \end{gathered} \tag{76}$$

where

$$\mathrm{d}\eta = \; \mathrm{d}\Sigma(\boldsymbol{x}) \; \sin\theta^\nu \mathrm{d}\theta^\nu \mathrm{d}\psi^\nu \; . \tag{77}$$

For given (θ, ψ), the phase Φ' is stationary with respect to the variables of integration η if

$$\partial_{\sigma_\mu} \Phi' = 0 \quad \text{and} \quad \partial_{(\theta^\nu, \psi^\nu)} \Phi' = 0 \; ; \tag{78}$$

here,

$$\partial_{\sigma_\mu} \Phi' = -\boldsymbol{\Gamma} \cdot \partial_{\sigma_\mu} \boldsymbol{x} \; , \tag{79}$$

while

$$\begin{aligned} \partial_{\theta^\nu} \Phi' &= [\hat{\gamma}^{(\hat{M})}(\boldsymbol{y}) - \hat{\gamma}^{(\hat{M})}(\boldsymbol{x})] \cdot \partial_{\theta^\nu} \boldsymbol{r} \; + [\tilde{\gamma}^{(\tilde{N})}(\boldsymbol{y}) - \tilde{\gamma}^{(\tilde{N})}(\boldsymbol{x})] \cdot \partial_{\theta^\nu} \boldsymbol{s} \; , \\ \partial_{\psi^\nu} \Phi' &= [\hat{\gamma}^{(\hat{M})}(\boldsymbol{y}) - \hat{\gamma}^{(\hat{M})}(\boldsymbol{x})] \cdot \partial_{\psi^\nu} \boldsymbol{r} + [\tilde{\gamma}^{(\tilde{N})}(\boldsymbol{y}) - \tilde{\gamma}^{(\tilde{N})}(\boldsymbol{x})] \cdot \partial_{\psi^\nu} \boldsymbol{s} \; . \end{aligned} \tag{80}$$

The stationary points are denoted as $\sigma_\mu^0 = \sigma_\mu^0(\boldsymbol{y})$ and induce the mapping $\boldsymbol{x}(\sigma_\mu^0, L)$ for any L. The solution of the first equation in (78) with (79) implies that the stationary migration dip, $\boldsymbol{\nu}^0$, must be parallel to the geological dip, $\boldsymbol{\nu}_\phi$, i.e.,

$$\boldsymbol{\nu}^0 = \pm \boldsymbol{\nu}_\phi \; ; \tag{81}$$

one solution, σ_μ^0, of the second equation is easy to identify:

$$\boldsymbol{x}(\sigma_\mu^0, L) = \boldsymbol{y} \; ;$$

at this value one expects the peak contribution to the resolved medium perturbation. At $\boldsymbol{x}(\sigma_\mu^0, L)$, given $\boldsymbol{\nu}^0$, (θ, ψ) will determine the stationary values of $(\boldsymbol{s}, \boldsymbol{r})$. We denote the stationary points by $\eta^0 = (\sigma_\mu^0, \boldsymbol{\nu}^0(\boldsymbol{x}(\sigma_\mu^0, L)))$, and the stationary point set by H^0, which contains at least two elements (cf. Eq.(81)).

Applying the four-dimensional stationary phase approximation to Eq.(76) amounts to

$$\begin{aligned} \langle (\mathbf{c}^{(1)})'(\phi(\boldsymbol{y}))\, |\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{y}} \rangle = \sum_{\eta^0 \in \mathrm{H}^0} \mathrm{Re} \sum_J \int_{S^2} \frac{1}{8\pi^3} \int_{\mathbf{R}} \int_{\mathbf{R}^+} \left(\frac{2\pi}{\omega}\right)^2 \\ \exp\left[i\omega\, \Phi'(\boldsymbol{y}, L, \eta^0, k_\mathrm{r}, \theta, \psi) + i\frac{\pi}{4}\, \mathrm{sig}(\nabla_\eta \nabla_\eta \Phi')^0 \right] \frac{a(\boldsymbol{y}, L, \eta^0, k_\mathrm{r}, \theta, \psi)}{\sqrt{|\det(\nabla_\eta \nabla_\eta \Phi')^0|}} \\ \chi_J(k_\mathrm{r}, \theta, \psi)\, h(\boldsymbol{y}, \boldsymbol{\nu}^0) \times \\ (\mathbf{c}^{(1)})'(\phi(\boldsymbol{x}(\sigma_\mu^0, L)))\, |\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{x}(\sigma_\mu^0, L)} \left. \frac{|\partial_{\sigma_1}\boldsymbol{x} \wedge \partial_{\sigma_2}\boldsymbol{x}|}{|\nabla_{\boldsymbol{x}}\phi|} \right|_{\boldsymbol{x}(\sigma_\mu^0, L)} \omega^2 \mathrm{d}\omega\, \mathrm{d}L\, \mathrm{d}\theta\, \mathrm{d}\psi \end{aligned} \tag{82}$$

in the absence of singularities. [Singularities require a separate analysis, which we will discuss in a separate paper.] Here, sig denotes the number of positive eigenvalues minus the number of negative eigenvalues of a matrix. In Eq.(82), k_r is the stretch of frequency with $|\boldsymbol{\Gamma}(\boldsymbol{y})|^0$, the norm of the gradient of travel time in case the migration dip is stationary.

In terms of the scattering coefficients (cf. Eq.(68)), for $\boldsymbol{x}(\sigma_\mu^0, L)$ near $\boldsymbol{y} \in \{\phi = L\}$, expression (82) reduces to

$$\begin{aligned} \langle (\mathbf{c}^{(1)})'(\phi(\boldsymbol{y}))\, |\nabla_{\boldsymbol{x}}\phi|_{\boldsymbol{y}} \rangle \simeq \sum_{\boldsymbol{\nu}^0 = \pm\boldsymbol{\nu}_\phi} \tfrac{1}{2} \sum_J \int_{S^2} \int_{\mathbf{R}} \\ \mathrm{Re}\, \frac{1}{\pi} \int_{\mathbf{R}^+} \exp\left[i\omega\, \Phi'(\boldsymbol{y}, L, \eta^0, k_\mathrm{r}, \theta, \psi) + i\frac{\pi}{4}\, \mathrm{sig}(\nabla_\eta \nabla_\eta \Phi')^0 \right] \chi_J(k_\mathrm{r}, \theta, \psi) \mathrm{d}k_\mathrm{r} \\ \times \frac{\mathbf{a}_R(\boldsymbol{y}, L, \eta^0, k_\mathrm{r}, \theta, \psi)\, R_L^{(\hat{N}\hat{M})}(\eta^0, k_\mathrm{r}, \theta, \psi)\, h(\boldsymbol{y}, \boldsymbol{\nu}^0)}{|\boldsymbol{\Gamma}(\boldsymbol{y})|^0 \sqrt{|\det(\nabla_\eta \nabla_\eta \Phi')^0|}} \\ \times \left. \frac{|\partial_{\sigma_1}\boldsymbol{x} \wedge \partial_{\sigma_2}\boldsymbol{x}|}{|\nabla_{\boldsymbol{x}}\phi|} \right|_{\boldsymbol{x}(\sigma_\mu^0, L)} \mathrm{d}L\, \mathrm{d}\theta\, \mathrm{d}\psi \; . \end{aligned} \tag{83}$$

Recognizing the bandlimited Dirac distribution,

$$I_{\Delta, J}(\boldsymbol{\nu}^0 \cdot (\boldsymbol{y} - \boldsymbol{x})) \equiv \frac{1}{\pi} \times \tag{84}$$

$$\int_{\mathbb{R}^+} \exp\left[i\omega\,\Phi'(\boldsymbol{y}, L, \eta^0, k_\mathrm{r}, \theta, \psi) + i\frac{\pi}{4}\,\mathrm{sig}(\nabla_\eta\nabla_\eta\Phi')^0\right]\,\chi_J(k_\mathrm{r}, \theta, \psi)\,\mathrm{d}k_\mathrm{r}\ ,$$

we find that the reconstruction amounts to a weighted integration of scattering coefficients,

$$\langle (\mathbf{c}^{(1)})'(\phi(\boldsymbol{y}))\,|\nabla_{\boldsymbol{x}}\phi|\boldsymbol{y}\rangle \simeq \sum_{\boldsymbol{\nu}^0 = \pm\boldsymbol{\nu}_\phi} \tfrac{1}{2}\sum_J \int_{E_\theta\times E_\psi}\int_{\mathbb{R}} \mathrm{Re}\ I_{\Delta,J}(\boldsymbol{\nu}^0\cdot(\boldsymbol{y}-\boldsymbol{x}))\,\frac{1}{|\nabla_{\boldsymbol{x}}\phi|}\bigg|_{\boldsymbol{x}(\sigma^0_\mu,\,L)} \times \frac{\mathbf{a}_R(\boldsymbol{y}, L, \eta^0, k_\mathrm{r}, \theta, \psi)\,R_L^{(\tilde{N}\hat{M})}(\eta^0, k_\mathrm{r}, \theta, \psi)\,h(\boldsymbol{y}, \boldsymbol{\nu}^0)}{|\boldsymbol{\Gamma}(\boldsymbol{y})|^0\sqrt{|\det(\nabla_\eta\nabla_\eta\Phi')^0|}} \times |\partial_{\sigma_1}\boldsymbol{x}\wedge\partial_{\sigma_2}\boldsymbol{x}|_{\boldsymbol{x}(\sigma^0_\mu,\,L)}\ \mathrm{d}L\ \mathrm{d}\theta\,\mathrm{d}\psi\ , \tag{85}$$

where we have accounted for the fact that the range of integration over θ, ψ will be limited by the acquisition geometry. The ranges are given by $E_\theta = E_\theta(\boldsymbol{\nu}^0)$, $E_\psi = E_\psi(\boldsymbol{\nu}^0, \theta)$.

In Section 9, we will evaluate the Hessian $\det(\nabla_\eta\nabla_\eta\Phi')^0$ and show that $\mathrm{sig}(\nabla_\eta\nabla_\eta\Phi')^0 = 0$ at $\boldsymbol{\nu}^0 = \pm\boldsymbol{\nu}_\phi$. Then, in case the partition is complete, we have

$$\sum_J \mathrm{Re}\ I_{\Delta,J}(\boldsymbol{\nu}^0\cdot(\boldsymbol{y}-\boldsymbol{x}))\,\frac{1}{|\nabla_{\boldsymbol{x}}\phi|} = \frac{1}{|\nabla_{\boldsymbol{x}}\phi|}\,\delta(\boldsymbol{\nu}^0\cdot(\boldsymbol{y}-\boldsymbol{x})) \simeq \delta(\phi(\boldsymbol{y}) - \phi(\boldsymbol{x}))\ . \tag{86}$$

Due to the point symmetry of the slowness surface at the image point $\boldsymbol{y}$, we can replace the summation $\sum_{\boldsymbol{\nu}^0 = \pm\boldsymbol{\nu}_\phi} \frac{1}{2}$ by the substitution $\boldsymbol{\nu}^0 = \boldsymbol{\nu}_\phi$.

8. GRT inversion of Kirchhoff data

Now, we will analyse the resolution operator from a different perspective. To accommodate for the non-linear reflection/transmission coefficients, we substitute our Kirchhoff-like approximation, a mixture of Eqs.(37) and (45), into the inversion formula Eq.(61). The result is an equation very similar to Eq.(85). We obtain

$$\langle (\mathbf{c}^{(1)})'(\phi(\boldsymbol{y}))\,|\nabla_{\boldsymbol{x}}\phi|\boldsymbol{y}\rangle \simeq \sum_{\boldsymbol{\nu}^0 = \pm\boldsymbol{\nu}_\phi} \tfrac{1}{2}\int_{E_\theta\times E_\psi}\int_{\mathbb{R}} \mathbf{r}_L^{(\tilde{N}\hat{M})}\,\mathrm{d}L\ \mathrm{d}\theta\,\mathrm{d}\psi\ , \tag{87}$$

where

$$\mathbf{r}_L^{(\tilde{N}\hat{M})} = \sum_J \mathrm{Re}\ I_{\Delta,J}(\boldsymbol{\nu}^0\cdot(\boldsymbol{y}-\boldsymbol{x}))\,\frac{1}{|\nabla_{\boldsymbol{x}}\phi|}\bigg|_{\boldsymbol{x}(\sigma^0_\mu,\,L)} \times \frac{\mathbf{a}_R(\boldsymbol{y}, L, \eta^0, k_\mathrm{r}, \theta, \psi)\,\mathrm{R}_L^{(\tilde{N}\hat{M})}(\eta^0, k_\mathrm{r}, \theta, \psi)\,h(\boldsymbol{y}, \boldsymbol{\nu}^0)}{|\boldsymbol{\Gamma}(\boldsymbol{y})|^0\sqrt{|\det(\nabla_\eta\nabla_\eta\Phi')^0|}} \times |\partial_{\sigma_1}\boldsymbol{x}\wedge\partial_{\sigma_2}\boldsymbol{x}|_{\boldsymbol{x}(\sigma^0_\mu,\,L)}\ . \tag{88}$$

Componentwise, the reflection/transmission coefficients can be identified, viz., by undoing the multiplications by $\mathbf{a}_R$.

9. Imaging reflectivity

9.1. The Hessian

In this section we will evaluate the Hessian of Φ' at a stationary point. First, note that $\partial_{\nu_\mu}\partial_{\nu_\nu}\Phi' = 0$ at the stationary point $\boldsymbol{x} = \boldsymbol{y}$ (then $\Phi \equiv 0$). Hence,

$$\det(\nabla_\eta \nabla_\eta \Phi')^0 = \left[\det(\nabla_{\boldsymbol{\nu}} \nabla_{\boldsymbol{\sigma}} \Phi')^0\right]^2 . \tag{89}$$

On the other hand, note that

$$\left(\partial_{\nu_\mu}\partial_{\sigma_\nu}\Phi'\right) = (\partial_{\nu_\mu}\boldsymbol{\Gamma}) \cdot (\partial_{\sigma_\nu}\boldsymbol{x}) . \tag{90}$$

This matrix can be written in the form

$$\begin{pmatrix} - & \partial_{\nu_1}\boldsymbol{\Gamma} & - \\ - & \partial_{\nu_2}\boldsymbol{\Gamma} & - \end{pmatrix} \begin{pmatrix} | & | \\ \partial_{\sigma_1}\boldsymbol{x} & \partial_{\sigma_2}\boldsymbol{x} \\ | & | \end{pmatrix} ,$$

hence, also,

$$|\boldsymbol{\Gamma}| \det[(\partial_{\boldsymbol{\nu}}\boldsymbol{\Gamma}) \cdot (\partial_{\boldsymbol{\sigma}}\boldsymbol{x})] = \det\left[\begin{pmatrix} - & \partial_{\nu_1}\boldsymbol{\Gamma} & - \\ - & \partial_{\nu_2}\boldsymbol{\Gamma} & - \\ - & \boldsymbol{\Gamma} & - \end{pmatrix} \begin{pmatrix} | & | & | \\ \partial_{\sigma_1}\boldsymbol{x} & \partial_{\sigma_2}\boldsymbol{x} & \boldsymbol{\nu}_\phi \\ | & | & | \end{pmatrix}\right] . \tag{91}$$

In this expression, the first matrix on the right-hand side can be identified as h (see Eq.(73)) and the second one with the Jacobian of the coordinate transformation Eq.(32). Hence, using Eq.(89), we arrive at the identity

$$|\boldsymbol{\Gamma}|^0 \sqrt{\det(\nabla_\eta \nabla_\eta \Phi')^0} = h(.,\boldsymbol{\nu}^0)\ |\partial_{\sigma_1}\boldsymbol{x} \wedge \partial_{\sigma_2}\boldsymbol{x}|_{\boldsymbol{x}(\sigma^0_\mu,\, L)} . \tag{92}$$

In view of Eq.(89), the positive and negative eigenvalues must come in pairs, so that $\operatorname{sig}(\nabla_\eta \nabla_\eta \Phi')^0$ must be equal to 0 or 4. On the other hand, we have intrinsically assumed that the gradient of two-way travel does not vanish, so that $h \not\equiv 0$. In accordance with Eq.(92), hence, the determinant cannot vanish. This implies that under continuous deformations of the interface and ray geometries, the signature of the Hessian cannot change from 0 to 4 or vice versa (this would require an eigenvalue to become zero). In the case of flat level surfaces, it can be shown that the signature equals zero, which now implies that

$$\operatorname{sig}(\nabla_\eta \nabla_\eta \Phi')^0 = 0$$

for any ϕ.

9.2. *The modified GRT inversion*

Substituting Eq.(92) into Eq.(88) now yields

$$\mathbf{r}_L^{(\tilde{N}\hat{M})} = \sum_J \operatorname{Re} I_{\Delta,J}(\boldsymbol{\nu}^0 \cdot (\boldsymbol{y} - \boldsymbol{x}))\, \frac{1}{|\nabla_{\boldsymbol{x}}\phi|}\bigg|_{\boldsymbol{x}(\sigma_\mu^0,\, L)} \times$$
$$\mathbf{a}_R(\boldsymbol{y}, L, \eta^0, k_\mathrm{r}, \theta, \psi)\, \mathsf{R}_L^{(\tilde{N}\hat{M})}(\eta^0, k_\mathrm{r}, \theta, \psi)\,, \tag{93}$$

which is the proper interpretation of the set of images created by the GRT inversion formula (61). To arrive at this expression, we could use that at the stationary dip $\boldsymbol{\nu}^0 = \boldsymbol{\nu}_\phi$, and we could set $(\boldsymbol{\nu} \cdot \boldsymbol{\nu}_\phi) = 1$ in Eq.(61) to begin with. Thus, a priori knowledge about the geological dip is not required.

In the stationary phase approximation, we can rewrite inversion formula (61) with Eq.(87) as

$$\int_{\mathbf{R}} \mathbf{r}_L^{(\tilde{N}\hat{M})}\, \mathrm{d}L \simeq \frac{1}{8\pi^2}\int_{S^2} \tag{94}$$
$$\Lambda_{\boldsymbol{y}}^{-1} \mathbf{w}^{(\tilde{N}\hat{M})}(\boldsymbol{y}, \tilde{\boldsymbol{\alpha}}^{(\tilde{N})}(\boldsymbol{y}), \hat{\boldsymbol{\alpha}}^{(\hat{M})}(\boldsymbol{y}))\, \partial_t u^{(\tilde{N}\hat{M})}(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{y})\, |\boldsymbol{\Gamma}|^4\, \frac{\partial(\tilde{\boldsymbol{\alpha}}, \hat{\boldsymbol{\alpha}})}{\partial(\boldsymbol{\nu}, \theta, \psi)}\bigg|_{\boldsymbol{y}} \mathrm{d}\boldsymbol{\nu}\,.$$

Here, we employ the mappings defined in Eq.(69).

To extract the reflection coefficient from $\mathbf{r}^{(\tilde{N}\hat{M})}$, one has to estimate the $\mathbf{a}_R$, which are functions of the stationary dip, scattering angle and azimuth. In the inversion procedure, we control the scattering angle and azimuth; in principle, we can estimate the geological dip from any of the images. With this estimate, the $\mathbf{a}_R$ can be evaluated. Now, note that (93) comprises a system of equations; from each equation, in principle, the reflection coefficient at specular can be determined. This redundancy can be employed to verify or improve the estimate of the stationary dip. The stationary dip also appears in the spectrum of the medium's perturbation; this is discussed in Appendix A.

On the other hand, by virtue of the stationary phase approximation, we can remove the AVA inversion nested in formula (61): consider the procedure

$$\int_{\mathbf{R}} \mathbf{r}_L^{(\tilde{N}\hat{M})}\, \mathrm{d}L \simeq \frac{1}{8\pi^2}\int_{S^2} \frac{|\boldsymbol{\Gamma}(\boldsymbol{y})|^m\, \partial_t u^{(\tilde{N}\hat{M})}(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{y})}{\left[\tilde{V}^{(\tilde{N})}(\boldsymbol{y})(\hat{V}^{(\hat{M})}(\boldsymbol{y}))^3\right]^{1/2}}\, \mathrm{d}\boldsymbol{\nu}\,; \tag{95}$$

then $\mathbf{a}_R$ must be replaced by the scalar quantity

$$\mathbf{a}_R(\boldsymbol{y}, \boldsymbol{x}, \Theta) \to \frac{A_u(\boldsymbol{x})}{A_u(\boldsymbol{y})}\, \hat{\xi}_p^{(\hat{M})}(\boldsymbol{r}, \boldsymbol{y})\hat{\xi}_p^{(\hat{M})}(\boldsymbol{r}, \boldsymbol{x})\, \tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s}, \boldsymbol{y})\tilde{\xi}_q^{(\tilde{N})}(\boldsymbol{s}, \boldsymbol{x})$$
$$\times\, |\boldsymbol{\Gamma}(\boldsymbol{y})|^{m-3}\, |\boldsymbol{\Gamma}(\boldsymbol{x})|\, \frac{(\boldsymbol{\nu} \cdot \boldsymbol{\nu}_\phi)_{\boldsymbol{x}}}{(\boldsymbol{\nu} \cdot \boldsymbol{\nu}_\phi)_{\boldsymbol{y}}}\,. \tag{96}$$

Its diagonal is given by

$$\mathbf{a}_R(\boldsymbol{y}, \boldsymbol{y}, \Theta) \to |\boldsymbol{\Gamma}(\boldsymbol{y})|^{m-2}\,. \tag{97}$$

By producing images both with $m = 4$ and $m = 3$, $|\Gamma(\boldsymbol{y})|$ at stationary can be imaged as well, viz., from the ratio of the images. Then the stationary dip is not required to find the reflection coefficient.

10. Discussion

We have shown, by carrying out a stationary phase resolution analysis, that it is feasible to extract information about the angular dependent reflection/transmission coefficients from a GRT-based migration/inversion. We did not have to linearise nor expand the coefficients. In fact, the outcome of the resolution analysis is a multiple set of images for the reflection/transmission coefficients for the available range of specular scattering angles. Any type of AVA analysis can then be applied to interpret those images. In the derivation we have made use of the fact that the surface integral representations are linear in the scattering coefficients; these coefficients reduce, at specular, to the reflection/transmission coefficients.

The GRT approach employs a somewhat unusual input of data, viz., via common (θ, ψ) gathers. The inversion formula reduces to a two-dimensional integration over migration dip. The (θ, ψ) sorting, however, varies with the image point. It bears resemblance with the sorting in common offset, though. The use of such a sorting, however, necessitates the calculation of an additional Jacobian.

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Appendix A. The spectrum of the medium perturbation

In principle, the geological dip can be directly estimated from an image. However, the geological dip is also hidden in the spectrum of the medium perturbation Eq.(33).

Setting (cf. Eq.(46))

$$z \parallel \boldsymbol{\nu}_\phi, \quad \{x_\mu\} \perp \boldsymbol{\nu}_\phi$$

at $\boldsymbol{y} \in \mathcal{D}$, the medium perturbation spectrum is of the form (cf. Eq.(33))

$$\begin{aligned}\widetilde{\partial_\phi \mathbf{c}^{(1)}}(\boldsymbol{k}) = &\int_{\mathbf{R}} \mathrm{d}L\, (\mathbf{c}^{(1)})'(L) \\ &\int_{\mathcal{D}} \delta(\phi(\boldsymbol{x}) - L) \exp\left[-i \left(k_\mu x_\mu + k_z(z\boldsymbol{y}(L) + z)\right)\right] \, \mathrm{d}\boldsymbol{x} \,, \end{aligned} \tag{A1}$$

where in fact $\boldsymbol{k} = \Theta' = k_r \boldsymbol{\nu}$. Near the level surface $\phi = L$, we employ the expansion

$$\phi(\boldsymbol{x}) = L + |\nabla_{\boldsymbol{x}}\phi| z + \tfrac{1}{2} x_\mu \phi_{\mu\nu} x_\nu \; ; \tag{A2}$$

we implicitly assume that the integral over L is windowed around $\boldsymbol{y}$ such that in the window or neighborhood $\phi_{\mu\nu}$ may depend on L but $\boldsymbol{\nu}_\phi$ does not. At the level surface on which $\boldsymbol{y}$ lies, we have $z_{\boldsymbol{y}} = 0$. Then,

$$\begin{aligned}
\partial_\phi \widetilde{\mathbf{c}^{(1)}}(\boldsymbol{k}) &\simeq \int_{\mathbf{R}} \mathrm{d}L \, (\mathbf{c}^{(1)})'(L) \exp[-ik_z z_{\boldsymbol{y}}(L)] \\
&\quad \int_{\mathcal{D}} \delta(|\nabla_{\boldsymbol{x}}\phi| z + \tfrac{1}{2} x_\mu \phi_{\mu\nu} x_\nu) \exp\left[-i\,(k_\mu x_\mu + k_z z)\right] \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}z \\
&= \int_{\mathbf{R}} \mathrm{d}L \, (\mathbf{c}^{(1)})'(L) \exp[-ik_z z_{\boldsymbol{y}}(L)] \\
&\quad \int_{\mathbf{R}^2} \exp\left[-i\left(k_\mu x_\mu - k_z \frac{x_\mu \phi_{\mu\nu} x_\nu}{2|\nabla_{\boldsymbol{x}}\phi|}\right)\right] \frac{\mathrm{d}x_1 \, \mathrm{d}x_2}{|\nabla_{\boldsymbol{x}}\phi|} \\
&= \int_{\mathbf{R}} \frac{2\pi \, \exp\left[\pi i \, \mathrm{sig}(k_z \phi_{\mu\nu})/4\right]}{|k_z| \sqrt{|\det(\phi_{\mu\nu}|}} \exp\left[ik_\mu \frac{|\nabla_{\boldsymbol{x}}\phi|}{k_z} \phi^{-1}_{\mu\nu} k_\nu\right] \times \\
&\quad (\mathbf{c}^{(1)})'(L) \exp[-ik_z z_{\boldsymbol{y}}(L)] \, \mathrm{d}L \; ,
\end{aligned} \tag{A3}$$

where sig, as in the main text, represents the sum of signs (± 1) of eigenvalues of $k_z \phi_{\mu\nu}$. Note that $k_\mu = 0$ corresponds with the geological dip direction.

If the level surfaces of ϕ were *flat* and $\boldsymbol{\nu}_\phi = \boldsymbol{\nu}^1$ fixed, we would get

$$\begin{aligned}
\partial_\phi \widetilde{\mathbf{c}^{(1)}}(\boldsymbol{k}) &= \delta(\boldsymbol{k} - (\boldsymbol{k} \cdot \boldsymbol{\nu}^1)\boldsymbol{\nu}^1) \, (\widetilde{\mathbf{c}^{(1)}})'(\boldsymbol{k} \cdot \boldsymbol{\nu}^1) \\
&= \frac{1}{k_r^2} \, \delta(\boldsymbol{\nu} - (\boldsymbol{\nu} \cdot \boldsymbol{\nu}^1)\boldsymbol{\nu}^1) \, (\widetilde{\mathbf{c}^{(1)}})'(k_r(\boldsymbol{\nu} \cdot \boldsymbol{\nu}^1)) \; ,
\end{aligned}$$

which, in the Radon domain, implies

$$\begin{aligned}
&\int_{\mathcal{D}} \partial_\phi \mathbf{c}^{(1)}(\phi(\boldsymbol{x})) \, \delta''(\boldsymbol{y} \cdot \boldsymbol{\nu} - \boldsymbol{x} \cdot \boldsymbol{\nu}) \, \mathrm{d}\boldsymbol{x} \\
&= -\frac{1}{\pi} \, \delta(\boldsymbol{\nu} - (\boldsymbol{\nu} \cdot \boldsymbol{\nu}^1)\boldsymbol{\nu}^1) \, \mathrm{Re} \int_{\mathbf{R}^+} (\widetilde{\mathbf{c}^{(1)}})'(k_r(\boldsymbol{\nu} \cdot \boldsymbol{\nu}^1)) \, \exp(ik_r\varphi) \, \mathrm{d}k_r \Big|_{\varphi = \boldsymbol{y} \cdot \boldsymbol{\nu}} \\
&= -\frac{1}{\pi} \, \delta(\boldsymbol{\nu} - (\boldsymbol{\nu} \cdot \boldsymbol{\nu}^1)\boldsymbol{\nu}^1) \, \mathrm{Re} \int_{\mathbf{R}^+} (\widetilde{\mathbf{c}^{(1)}})'(k_r) \, \exp(ik_r\varphi) \, \mathrm{d}k_r \Big|_{\varphi = \boldsymbol{y} \cdot \boldsymbol{\nu}^1} \; .
\end{aligned} \tag{A4}$$

This formula shows that the GRT algorithm can reveal the geological dip explicitly. We assumed a proper coordinate system on S^2, such that

$$\int_{S^2} \delta(\boldsymbol{\nu} - (\boldsymbol{\nu} \cdot \boldsymbol{\nu}^1)\boldsymbol{\nu}^1) \, \mathrm{d}\boldsymbol{\nu} = 1 \; .$$

References

[1] Geoltrain S. and Chovet E., 1991, Automatic association of kinematic information to prestack images, *61st Ann. Int. Mtg. Soc. Explor. Geophys.*, Expanded Abstracts, pp.890-892.

[2] Lumley D.E., 1993, Angle-dependent reflectivity estimation, *63rd Ann. Int. Mtg. Soc. Expl. Geophys.*, Expanded Abstracts, pp.746-749.

[3] Bleistein N., Cohen J.K. and Stockwell J., 1996, *Mathematics of multidimensional seismic inversion*, in preparation.

[4] Norton S.G. and Linzer M., 1981, Ultrasonic scattering potential imaging in three dimensions: exact inverse scattering solutions for plane, cylindrical, and spherical apertures, *IEEE Trans. on Biomedical Engineering* **BME-28**, 202-220.

[5] Beylkin G., 1982, *Generalized Radon Transform and its applications* (Ph.D. Thesis, New York University).

[6] Beylkin G., 1984, The inversion problem and applications of the generalized Radon Transform, *Comm. Pure Appl. Math.* **37**, 579-599.

[7] Beylkin G., 1985, Imaging of discontinuities in the inverse scattering problem by inversion of a causal generalized Radon transform, *J. of Math. Phys.* **26**, 99-108.

[8] Beylkin G., 1985, Reconstructing discontinuities in multidimensional inverse scattering problems: smooth errors vs small errors, *Applied Optics* **24**, 4086-4088.

[9] Miller D.E., Oristaglio M. and Beylkin G., 1984, A new formalism and an old heuristic for seismic migration, *54th Ann. Int. Mtg. Soc. Explor. Geophys.*, Expanded Abstracts, pp.704-707.

[10] Miller D.E., Oristaglio M. and Beylkin G., 1987, A new slant on seismic imaging: migration and integral geometry, *Geophysics* **52**, 943-964.

[11] Beylkin G., Oristaglio M. and Miller D.E., 1985, Spatial resolution of migration algorithms, in: Berkhout A.J., Ridder J. and van der Waal L.F., *Acoustical Imaging* **14**, Plenum Pub. Co., pp.155-167.

[12] Rakesh, 1988, A linearised inverse problem for the wave equation, *Comm. in Part. Diff. Eqs.* **13**, 573-601.

[13] Beylkin G. and Burridge R., 1990, Linearized inverse scattering problems in acoustics and elasticity, *Wave Motion* **12**, 15-52.

[14] de Hoop M.V., Burridge R., Spencer C. and Miller D.E., 1994, Generalized Radon Transform/amplitude versus angle (GRT/AVA) migration/inversion in anisotropic media, *SPIE proceedings* **2301**, pp.15-27.

[15] Spencer C. and de Hoop M.V., 1995, Linearity, resolution, and covariance in GRT inversions for anisotropic elastic moduli, *SPIE proceedings* **2571**, pp.110-122.

[16] Cohen J.K. and Bleistein N., 1977, An inverse method for determining small variations in propagation speed, *SIAM J. Appl. Math.* **32** 784-799.

[17] Bleistein N. and Cohen J.K., 1979, Velocity inversion for acoustic waves, *Geophysics* **44** 1077-1087.

[18] Bleistein N., 1987, On imaging of reflectors in the earth, *Geophysics* **52**, 931-942.

[19] Bleistein N., 1987, Kirchhoff inversion for reflector imaging and sound speed and density variations, Worthington, M., Ed., *Deconvolution and inversion: 1986 EAEG/SEG Workshop (Rome, Italy)* (Oxford: Blackwell Scientific Publishers) pp.305-320.

[20] de Hoop M.V. and Spencer C., 1996, Quasi Monte-Carlo integration over $S^2 \times S^2$ for migration $\times$ inversion, to appear in *Inverse Problems*.

[21] Kendall J-M., Guest W.S. and Thomson C.J., 1992, Ray-theory Green's function reciprocity and ray-centred coordinates in anisotropic media, *Geoph. J. Int.* **108**, 364-371.

[22] Hörmander L., 1983, *The analysis of linear partial differential operators III* (Berlin: Springer-Verlag) Chapter XXI.

Figure captions

Figure 1. Source-receiver ray geometry (S^2 = unit sphere).

Figure 2. Source-receiver ray polarizations.

Figure 3. Micro-local medium perturbation.